

Differentiability & Derivatives

ISI & CMI Crash Course 2024

lec 1

2024-04-09

★ Introduction

○ First Principles.

- Defⁿ: Let $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R}$. Pick $x_0 \in U$. $\exists \neq l \in \mathbb{R}$

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = l$$

Then we say f is differentiable @ $x_0 \in U$, and has derivative l @ x_0 , written as: $f'(x_0) = l$

- Eg: Differentiate $f(x) = \sin(x)$ using first principles.

$$\lim_{x \rightarrow x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{\sin\left(\frac{x-x_0}{2}\right) \cos\left(\frac{x+x_0}{2}\right)}{x - x_0}$$

$$= \lim_{x \rightarrow x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \cdot \cos\left(\frac{x+x_0}{2}\right)$$

$$= \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \cdot \lim_{x \rightarrow x_0} \cos\left(\frac{x+x_0}{2}\right) = \cos(x_0)$$

by continuity of $\cos(\cdot)$

- Result: $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $U \ni x_0$ f differentiable @ x_0
 $\Rightarrow f$ continuous @ x_0

Proof: $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = l \in \mathbb{R}$

$$\lim_{x \rightarrow x_0} (x - x_0) = 0$$

Multiplying

$$\begin{aligned} & \frac{f(x) - f(x_0)}{x - x_0} \rightarrow 0 \\ \Leftrightarrow & f(x) \rightarrow f(x_0) \end{aligned}$$

⑥ Left-Hand & Right-Hand Derivative

- $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be left-differentiable @ $x_0 \in U \stackrel{\text{open}}{=} (-\infty, x_0]$ if $\exists l \in \mathbb{R}$

$$\lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = l$$

where l is called the left-hand / left derivative @ x_0

$$\equiv \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = l$$

$$\equiv \lim_{h \downarrow 0} \frac{f(x_0 - h) - f(x_0)}{-h} = l$$

} all are equivalent.

- (similar for right-hand derivative)

$$\lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h}, \lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = l$$

- (usually denoted by f'_L, f'_R or f'_-, f'_+)

- Results: Prove these yourself. (new)

- i) Left differentiable $\Rightarrow$ left continuous
- ii) Right differentiable $\Rightarrow$ right continuous
- iii) Both L, R differentiable & LHD = RHD $\Rightarrow$ differentiable $\Rightarrow$ cts.
- iv) Both L, R differentiable & LHD $\neq$ RHD $\Rightarrow$ * cts but not differentiable
(not really, some more assumptions are needed) but in practice almost always true.

Essentially, U shouldn't have a hole @ $x_0 \dots$

- Warning

$$\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0-h)}{2h} \text{ exists } \nRightarrow f \text{ differentiable @ } x_0.$$

Go doesn't imply f is cts @ x_0 . Take: $f(x) = \begin{cases} x^3, & x \neq 0 \\ 1, & x = 0 \end{cases}$

- ⊙ Algebra of Differentiable Functions

- $\forall f$ differentiable, define $Df : \begin{matrix} x \mapsto f'(x) \\ U \rightarrow \mathbb{R} \end{matrix}$
on $U \subseteq \mathbb{R}$
open

- Suppose $f, g : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $\lambda \in \mathbb{R}$:
open

add'n
+ mult'n
+ scaling
= algebra

$$\left. \begin{aligned} D(f + \lambda g) &= Df + \lambda Dg \\ D(fg) &= fDg + gDf \end{aligned} \right\} \begin{array}{l} \text{as functions on } U. \\ \text{(taking values in } \mathbb{R}) \end{array}$$

If g is non-vanishing on U , then...

$$D\left(\frac{f}{g}\right) = \frac{gDf - fDg}{g^2}$$

- Chain Rule: $D(f \circ g) = (Df \circ g)(Dg)$

- Differentiation of Inverses: $f \circ f^{-1} \equiv \text{Id}$, $\text{Id}: x \mapsto x$

$$\Rightarrow \underbrace{[(Df) \circ f^{-1}]}_{1: x \mapsto 1} (Df^{-1}) \equiv 1 \Rightarrow Df^{-1} \equiv \frac{1}{[(Df) \circ f^{-1}]}$$

⊛ Higher-Order Derivatives.

⊙ Introduction (theory is exactly the same)

- Write: $f^{(0)} \equiv f$, $f^{(r)} \equiv Df^{(r-1)} \quad \forall r=1, \dots, N$

--- where $N \mid f^{(N)}$ not differentiable on U .

- D is linear, hence so is $D^2, D^3, \dots$ higher order diffⁿ op.

$$Df \equiv f^{(1)}, D^2 f \equiv D(Df) \equiv f^{(2)}, \dots$$

- Product Rule for Higher Order Derivatives.

$$D^n[fg] \equiv \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)} \quad \rightarrow \text{kinda like BT for } D$$

- Ex-) Compute the 3rd derivative of:

$$\underbrace{(x \sin(x) + \cos(x))}_f \underbrace{(e^x - \ln x + x^3)}_g$$

$f^{(0)} = x \sin x + \cos x$	$g^{(0)} = e^x - \ln x + x^3$
$f^{(1)} = \cancel{\sin x} + x \cos x - \cancel{\sin x}$	$g^{(1)} = e^x - \frac{1}{x} + 3x^2$
$f^{(2)} = -x \sin x + \cos x$	$g^{(2)} = e^x + \frac{1}{x^2} + 6x$
$f^{(3)} = -\sin x - x \cos x - \sin x$	$g^{(3)} = e^x - \frac{2}{x^3} + 6$

— $\leadsto \binom{3}{0} \binom{3}{3} = 1$ weightage

— $\leadsto \binom{3}{1} \binom{3}{2} = 3$ weightage.

① Classes of Functions based on Differentiability

- Let $U \subseteq \mathbb{R}$. Define

$$C^r(U) := \{ f: U \rightarrow \mathbb{R} \mid f^{(r)} \text{ exists \& is cts on } U \}$$

- Notice, $\because$ differentiability $\Rightarrow$ continuity,
 $C^0(U) \supseteq C^1(U) \supseteq C^2(U) \supseteq \dots \supseteq C^r(U) \supseteq \dots$

- Define: $C^\infty(U) = \bigcap_{r=1}^{\infty} C^r(U)$ [$C^r(U) \downarrow C^\infty(U)$]
define

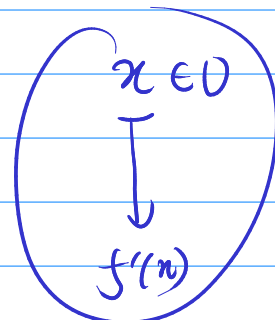
$\hookrightarrow$ functions which are continuously differentiable an arbitrary number of times.

- We know $C^0(U)$ is an algebra. It can be shown using standard results, $C^r(U)$ is an algebra $\forall r=1, 2, \dots, \infty$
 (Remember: algebra = add'n + mult'n + scaling of elements)

- $C^\omega(U) \subsetneq C^\infty(U)$ $\rightarrow$ all real analytic functions.
 $\Downarrow$ We will come back to this in Lec 4. [ω here represents the smallest infinite ordinal]

② What exactly is "D"?

- $D: \begin{matrix} f \\ C^k(U) \end{matrix} \mapsto \begin{matrix} Df \\ C^{k-1}(U) \end{matrix} =$



and D is linear. $\therefore D \in \mathcal{L}(C^k(U), C^{k-1}(U))$

- $\therefore$ $(?) \equiv D$ (for finite dimensions) $\hookrightarrow$ matrices $\cong$ linear transforms.
 $\hookrightarrow$ Jacobians (and higher orders are Hessians, etc.)

⊙ The Curious Case of $|x|^p \sin(\frac{1}{x})$

- Define $f_p(x) = \begin{cases} 0 & x=0 \\ |x|^p \sin(\frac{1}{x}) & x \neq 0 \end{cases}$

We have already established that f_p is cts. @ 0 $\forall p > 0$.

- Formally, $\frac{d}{dx} \left(|x|^p \sin(\frac{1}{x}) \right)$
$$= p|x|^{p-1} \operatorname{sgn}(x) \sin(\frac{1}{x}) + |x|^p \left(-\frac{1}{x^2}\right) \cos(\frac{1}{x})$$
$$= p|x|^{p-1} \operatorname{sgn}(x) \sin(\frac{1}{x}) - |x|^{p-2} \cos(\frac{1}{x})$$

This is cts. @ $x=0$ iff $p > 1$ HW

$\therefore f_p$ is C^1 iff $p > 1$.

- Result $\forall k \in \mathbb{N}_0$, f_p is $C^k \Leftrightarrow p > k$

@ $p=k$, the formal k^{th} derivative has an $x^0 \sin(\frac{1}{x})$ term.

⊙ Relation with Graph Sketching

- Smoothness $\Rightarrow C^1$ (or C^2)  "Smooth turns"
- Sharp corner $\Rightarrow$ non-differentiable
- Maxima/minima/inflection point $\Leftarrow$ derivative 0 (if cts.)
 - Second derivative $> 0 \Leftrightarrow$ minima
 - " " $< 0 \Leftrightarrow$ maxima
 - Inflection points are weird, can have all derivatives 0, even.
- Convex $\Leftarrow$ second derivative > 0 ; concave $\Leftarrow$ second derivative < 0

★ Mean Value Theorems

① Rolle's Theorem

- Fermat's Theorem: $f: (a, b) \rightarrow \mathbb{R}$ has a local extremum at $x_0 \in (a, b)$. If f is differentiable @ x_0 , $f'(x_0) = 0$.

Suppose that x_0 is a local maximum (a similar proof applies if x_0 is a local minimum). Then there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and such that we have $f(x_0) \geq f(x)$ for all x with $|x - x_0| < \delta$. Hence for any $h \in (0, \delta)$ we have

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

Since the limit of this ratio as h gets close to 0 from above exists and is equal to $f'(x_0)$ we conclude that $f'(x_0) \leq 0$. On the other hand, for $h \in (-\delta, 0)$ we notice that

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

but again the limit as h gets close to 0 from below exists and is equal to $f'(x_0)$ so we also have $f'(x_0) \geq 0$.

Hence we conclude that $f'(x_0) = 0$.

- Rolle's Theorem: $f: [a, b] \rightarrow \mathbb{R}$ continuous, f is differentiable on (a, b) . $f(a) = f(b)$
Then $\exists x_0 \in (a, b) \mid f'(x_0) = 0$.

→ Proof:

Case i): f is a constant function. (Trivial, choose any x_0)

Case ii): f is non-constant

$\therefore f$ is continuous on a compact set, attains an absolute maximum and minimum.

$\therefore f$ is non-constant, at least one of them is attained in (a, b) , not at endpoints. (else $\max = \min \Rightarrow f_{\text{const}}$)

Say, this happens @ $x_0 \in (a, b)$. Fix any $\delta > 0$.

Now, $\forall x \in [a, b] \supseteq (x_0 - \delta, x_0 + \delta)$ $f(x) \leq f(x_0)$
 $\Rightarrow x_0$ is a local maximum of f .

By Fermat's Theorem, $f'(x_0) = 0$

② Mean Value Theorem

• Cauchy's Mean Value Theorem

$f, g: [a, b] \rightarrow \mathbb{R}$ continuous ^{on $[a, b]$} & differentiable on (a, b)
Then $\exists x_0 \in (a, b) \mid$

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{assuming } g(b) \neq g(a)$$

Proof: Apply Rolle's Theorem to

$$\textcircled{\text{HW}} \quad h(x) := f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$$

• Lagrange's Mean Value Theorem

$f: [a, b] \rightarrow \mathbb{R}$ continuous, f differentiable on (a, b)
Then $\exists x_0 \in (a, b) \mid$

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

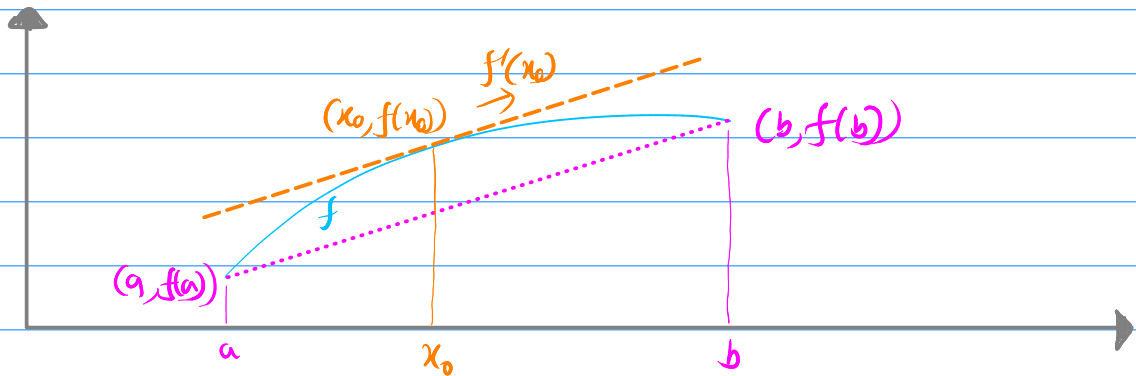
Proof: (special case of CMVT w/ $g: x \mapsto x$).

• Geometric Interpretation of LMVT

Now, $\frac{f(b) - f(a)}{b - a}$ $\leftarrow$ slope of secant between $(a, f(a))$ & $(b, f(b))$

& $f'(x)$ denotes slope of tangent to f @ $(x, f(x))$

$\therefore$ LMVT essentially says $\exists$ a tangent $\parallel$ to the secant joining endpoints.



Q

6. Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all x, y in $\mathbb{R}$.
7. Use the Mean Value Theorem to prove that $(x - 1)/x < \ln x < x - 1$ for $x > 1$. [Hint: Use the fact that $D \ln x = 1/x$ for $x > 0$.]
8. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) . Show that if $\lim_{x \rightarrow a} f'(x) = A$, then $f'(a)$ exists and equals A . [Hint: Use the definition of $f'(a)$ and the Mean Value Theorem.]

(Source: Bartle-Shurubert pp. 179)

Qns 6)

Pick any $x, y \in \mathbb{R} \mid x < y$.

Using LMVT, $\exists c \in (x, y) \mid \frac{\sin x - \sin y}{x - y} = \cos(c)$

Now, $|\cos(c)| \leq 1 \quad \forall c \in \mathbb{R} \Rightarrow \left| \frac{\sin x - \sin y}{x - y} \right| \leq 1$

$$\Rightarrow |\sin x - \sin y| \leq |x - y|$$

Ans 7) Pick any $x > 1$. By LMVT, $\exists c \in (1, x) \mid$

$$\frac{\ln x - \ln 1}{x-1} \overset{0}{=} D \ln x|_x = \frac{1}{x}$$

Now, $c \in (1, x) \Rightarrow \frac{1}{x} \in (\frac{1}{x}, 1)$

$$\Rightarrow \frac{1}{x} < \frac{\ln x}{x-1} < 1$$

$$\Rightarrow \frac{x-1}{x} < \ln x < x-1 \quad \blacksquare$$

Ans. 8) $\lim_{x \rightarrow a} f'(x) = A$

$$\Rightarrow \lim_{x \rightarrow a} \lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = A$$

$$\Rightarrow \lim_{y \rightarrow a} \lim_{x \rightarrow a} \frac{f(y) - f(x)}{y - x} = A \quad \left[\begin{array}{l} \text{By an interchange} \\ \text{of limits.} \end{array} \right]$$

$$\Rightarrow \lim_{y \rightarrow a} \frac{f(y) - f(a)}{y - a} = A \Rightarrow f'(a) = A \quad \blacksquare$$

[INTENDED SOLUTION]

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

By LMVT, $\exists \xi \in (a, x) \mid \frac{f(x) - f(a)}{x - a} = f'(\xi)$.

Also, as $x \rightarrow a$, $\xi \rightarrow a$.

$$\therefore f'(a) = \lim_{\xi \rightarrow a} f'(\xi) = A \quad (\text{given}) \quad \blacksquare$$

everything is nice
here, ignoring
technicalities

① Intermediate Value Property of Derivatives

$f: I = [a, b] \rightarrow \mathbb{R}$ differentiable on I .

Then, f' has IVP on I , even if it is not continuous!

- Lemma Let $f: I = [a, b] \rightarrow \mathbb{R}$, f differentiable @ $c \in I$.

$$a) \quad f'(c) > 0 \Rightarrow \exists \delta > 0 \mid \forall x \in (c, c + \delta) \quad f(x) > f(c)$$

$$b) \quad f'(c) < 0 \Rightarrow \exists \delta > 0 \mid \forall x \in (c - \delta, c) \quad f(x) > f(c)$$

new
first
week

(Proof is available in Bartle-Sharbert.)

- Darboux's Theorem

$f: I = [a, b] \rightarrow \mathbb{R}$ differentiable on I .

Then, if $k \in (f'(a) \wedge f'(b), f'(a) \vee f'(b))$
then $\exists c \in (a, b) \mid f'(c) = k$.

Essentially f' has IVP.

Proof: Suppose $f'(a) < k < f'(b)$.

$$g(x) := kx - f(x) \leftarrow \text{continuous, hence attains maxima on } [a, b]$$

$$\Rightarrow g'(x) = k - f'(x)$$

$$\begin{aligned} \hookrightarrow g'(a) &= k - f'(a) > 0 \Rightarrow \text{no maxima of } g \text{ @ } a \\ \hookrightarrow g'(b) &= k - f'(b) < 0 \Rightarrow \text{no maxima of } g \text{ @ } b \end{aligned}$$

$\Rightarrow \exists c \in (a, b)$ where g attains maxima (by above lemma)

$$g'(c) = k - f'(c) = 0 \Rightarrow f'(c) = k$$

★ L'Hôpital's Rules

⊙ The Gist

- If a limit is of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, where the limit in question is $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$, then given both f & g are differentiable in an nbd around a , $(a-\delta, a+\delta) \ni a$,

$$\& \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} =: L \in \overline{\mathbb{R}} = [-\infty, \infty] \leftarrow \begin{array}{l} \text{includes } \pm\infty \\ \text{ } \mathbb{R} \text{ "closure"} \end{array}$$

Then: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$ || Result holds analogously for $x \rightarrow a^+$, a^- etc.

- Proof: makes use of CMVT on $(a \pm \delta)$; easier for $0/0$ case, use CMVT and send $\delta \rightarrow 0$.

Ⓐ (Bartle-Shorbut Exc for §6.3)

6. Evaluate the following limits.

(a) $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$ 2

(b) $\lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^4}$ $\frac{1}{3}$

7. Evaluate the following limits, where the domain of the quotient is as indicated.

(a) $\lim_{x \rightarrow 0^+} \frac{\ln(x+1)}{\sin x}$ $(0, \pi/2)$, 1

(b) $\lim_{x \rightarrow 0^+} \frac{\tan x}{x}$ $(0, \pi/2)$, 1

(c) $\lim_{x \rightarrow 0^+} \frac{\ln \cos x}{x}$ $(0, \pi/2)$, 0

(d) $\lim_{x \rightarrow 0^+} \frac{\tan x - x}{x^3}$ $(0, \pi/2)$, $\frac{1}{3}$

8. Evaluate the following limits:

(a) $\lim_{x \rightarrow 0} \frac{\text{Arctan } x}{x}$ $(-\infty, \infty)$, 1

(b) $\lim_{x \rightarrow 0} \frac{1}{x(\ln x)^2}$ $(0, 1)$, ∞

(c) $\lim_{x \rightarrow 0^+} x^3 \ln x$ $(0, \infty)$, 0

(d) $\lim_{x \rightarrow \infty} \frac{x^3}{e^3}$ $(0, \infty)$, ∞

Q

12. Let f be differentiable on $(0, \infty)$ and suppose that $\lim_{x \rightarrow \infty} (f(x) + f'(x)) = L$. Show that $\lim_{x \rightarrow \infty} f(x) = L$ and $\lim_{x \rightarrow \infty} f'(x) = 0$. [Hint: $f(x) = e^x f(x)/e^x$.]

13. Try to use L'Hospital's Rule to find the limit of $\frac{\tan x}{\sec x}$ as $x \rightarrow (\pi/2)^-$. Then evaluate directly by changing to sines and cosines.

14. Show that if $c > 0$, then $\lim_{x \rightarrow c} \frac{x^c - c^x}{x^x - c^c} = \frac{1 - \ln c}{1 + \ln c}$.

⑫ $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{e^x f(x)}{e^x} = \lim_{x \rightarrow \infty} \frac{e^x (f(x) + f'(x))}{e^x}$ [L'Hôpital's Rule]

New subtract $\lim_{x \rightarrow \infty} f(x) = L \rightarrow L$ (given)

⑭ $\lim_{x \rightarrow c} \frac{x^c - c^x}{x^x - c^c} = \lim_{x \rightarrow c} \frac{cx^{c-1} - c^x \ln c}{x^x(1 + \ln x)} = \frac{e^c(1 - \ln c)}{e^c(1 + \ln c)}$
remember this!

Q (Doubt by Subhoshree)

i) $\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos 2x}}{\sqrt{2}x} = \lim_{x \rightarrow 0} \frac{\sqrt{2 \sin^2 x}}{\sqrt{2}x}$

$= \lim_{x \rightarrow 0} \frac{|\sin x|}{x} \leftarrow \text{doesn't exist} \begin{cases} \text{LHL} = -1 \\ \text{RHL} = +1 \end{cases}$ HWO check.

ii) $\lim_{x \rightarrow 3} \frac{x+2}{x-3} \leftarrow \text{doesn't exist} \begin{cases} \text{LHL} = -\infty \\ \text{RHL} = +\infty \end{cases} \left(\sim \frac{1}{x} \right)$

⑥ (Doubt by Simjan; CMI 2019)

Consider $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined as follows:

$$f(a, b) := \lim_{n \rightarrow \infty} \frac{1}{n} \log_e (e^{na} + e^{nb}) = \frac{a+b}{2} + \frac{|a-b|}{2}$$

Which of these statements are true?

- ✗ (a) f is not onto i.e. the range of f is not all of $\mathbb{R}$. $f(a, a) = a$, onto
- ✓ (b) For every a the function $g(x) = f(a, x)$ is continuous everywhere.
- ✗ (c) For every b the function $h(x) = f(x, b)$ is differentiable everywhere.
- ✓ (d) We have $f(0, x) = x$ for all $x \geq 0$

Observe: $f(a, b) = f(b, a) \quad \forall a, b$

$$f(x, a) = f(a, x) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln(e^{na} + e^{nx})$$

$$= \lim_{n \rightarrow \infty} \frac{na + \ln(1 + e^{n(x-a)})}{n}$$

$$= a + \lim_{n \rightarrow \infty} \frac{1}{n} \ln(1 + e^{n(x-a)})$$

$$= a + \lim_{n \rightarrow \infty} \frac{(x-a) e^{n(x-a)}}{1 + e^{n(x-a)}} \quad \left[\text{L'Hôpital's Rule} \right]$$

$$= a + (x-a) \lim_{n \rightarrow \infty} \frac{1}{1 + e^{-n(x-a)}} = L$$

If $x > a$, then as $n \rightarrow \infty$, $-n(x-a) \rightarrow -\infty$, $L = 1$
 $x < a$, then as $n \rightarrow \infty$, $-n(x-a) \rightarrow \infty$, $L = 0$
 $x = a$, then $L = 1/2$

$$\therefore f(x, a) = f(a, x) = a + (x-a) \left[\frac{1 + \operatorname{sgn}(x-a)}{2} \right]$$

$$= \frac{x+a}{2} + \frac{|x-a|}{2} \leftarrow \text{✗, (b) ✗, (d) ✓}$$

★ Taylor's Theorem and Expansion

◎ Taylor Polynomials

- Consider $U \subseteq \mathbb{R}$, $f: U \rightarrow \mathbb{R} \in C^k(U)$. $U \ni x_0$
Then $\forall n = 0, 1, \dots, k$

$$P_n: x \mapsto \sum_{j=0}^n \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j$$

n^{th} degree Taylor polynomial of f around x_0
order about

- Observe: $\left. \begin{matrix} f^{(0)}, f^{(1)}, \dots, f^{(k)} \\ P_n^{(0)}, P_n^{(1)}, \dots, P_n^{(k)} \end{matrix} \right\} \text{agree (correspondingly)} \\ \text{ @ } x_0$

$$\text{i.e. } \forall j = 0, 1, \dots, k \quad f^{(j)}(x_0) = P_n^{(j)}(x_0) = P_n^{(j)}(x_0)$$

Because of continuity, they are "close" in a nbd of x_0 .

- Define: $R_n := f - P_n \quad \forall n = 0, 1, \dots, k$

◎ Taylor's Theorem

- Suppose $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R} \in C^k(U)$; $U \ni x_0$.
Let P_n be the n^{th} degree Taylor polynomial of f around x_0
 $R_n := f - P_n$.

If f is $(n+1)$ -times differentiable (not necessarily C^{n+1}),
then $\forall x \in U \setminus \{x_0\}$, $\exists \xi \in (x \wedge x_0, x \vee x_0)$ |

$$R_k(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}$$

← Lagrange / derivative form of remainder

This can be proven inductively using LMVT (for $k=1$, $\equiv$ LMVT)

- Essentially:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-x_0)^{k+1}$$

... for some ξ b/w x and x_0 .

- If f is C^∞ , then we can extend this process infinitely

Under some* conditions, $\forall x \in U$ $P_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$
 $\hookrightarrow$ analytic functions $R_n(x) \rightarrow 0$

Hence, $\{P_n\}$ approximates f (under some conditions)

⊙ Taylor Series & Expansion

- Suppose $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ C^∞ . $U \ni x_0$.

$\therefore$ The series $\left\{ \sum_{j=0}^n \frac{f^{(j)}(x_0)}{j!} (x-x_0)^j \right\}_{n \in \mathbb{N}}$ is a power series defined on U .
 $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$ This is called the Taylor series of f around x_0 .

- For $x_0=0$, we call it the Maclaurin series of f .

- Now, the power series is defined $\forall x \in U$.
However, it is not necessary that it actually converge $\forall x \in U$.

↳ Can view the power series as a "function"

$$\text{Pos: } x \mapsto \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

$$x \mapsto \text{Series}(x)$$



may or may not converge.

See this as an expression, may not have a value as it may not converge

- Defⁿ: Let $\sum_{n=0}^{\infty} a_n (x-c)^n$ be a power series.

$$r := \sup \left\{ \delta \mid |x-c| < \delta \Rightarrow \sum_{n=0}^{\infty} a_n (x-c)^n \text{ converges} \right\}$$

r is called the radius of convergence of the power series

- Result: Let S be a power series w/ radius --- r .
centered @ c

$$\therefore S \text{ converges @ } x \quad \forall x \in (c \pm r)$$

NOTE: No guarantees @ boundaries: $\{c \pm r\}$

- For any Taylor series, $r > 0 \Rightarrow R_n(x) \rightarrow 0 \quad \forall x \in U$.

Hence, we have a dichotomy:

$$r = \begin{cases} 0 \\ \infty \end{cases}$$

o/w

$$f \in C^{\omega}(U)$$

defined this way
 f is real analytic on U .

Notice $C^{\omega} \subseteq C^{\infty}$

◉ In Practice...

- First most functions that we deal with are analytic (C^∞) (except at discontinuities etc). Essentially, if its C^∞ , it is almost always analytic.

- Special Case: Consider $f(x) = e^{-1/x} \mathbb{1}(x > 0)$.

(on check: f is $C^\infty(\mathbb{R})$)

HW tried in class

However, $\forall n \in \mathbb{N}_0, f^{(n)}(0) = 0$

HW

$\Rightarrow$ McLaurin series of $f \equiv 0 \neq f$.

$\therefore f$ is non-analytic but C^∞ .

$\circ$ Frechét PMFs ($\alpha > 0$)

- Special Case: $f_\alpha(x) = e^{-x^\alpha} \mathbb{1}(x > 0)$ (similar)

- Taylor as Repeated L'Hôpital.

In case you have a $0/0$ form limit which requires repeated L'Hôpital, Taylor series will do it in one step.

$$\text{Eg: } \lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^3} \quad \begin{matrix} x^2 \rightarrow 0(x^2) \\ \cancel{x} \end{matrix}$$

$$\circ = \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{-\sin x}{6x} = \lim_{x \rightarrow 0} \frac{-\cos x}{6} = -\frac{1}{6}$$

• In Subjective problems

In subjective questions, proving a f^n to be analytic is way too tough. Hence, avoid using the full expansion & instead work w/ P_n & Q_n .

Q

16. Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \rightarrow \mathbb{R}$ be differentiable on I , and suppose $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Give an example where this limit exists, but the function does not have a second derivative at a .

(Ans)

Trivial using L'Hôpital on h .

$$f''(a) = \lim_{h \rightarrow 0} \frac{f'(a+\frac{h}{2}) - f'(a-\frac{h}{2})}{h}$$

($\because$ this exists)

$$= \lim_{(h,k) \rightarrow (0,0)} \frac{f(a+\frac{h}{2}+\frac{k}{2}) - f(a+\frac{h}{2}-\frac{k}{2}) - f(a-\frac{h}{2}+\frac{k}{2}) + f(a-\frac{h}{2}-\frac{k}{2})}{hk}$$

double limit

Take the sequences along $h=k$

$$= \lim_{h \rightarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Eg.: Try $f(x) = x|x|$ @ $a=0$. $\leftarrow$ not twice diff @ 0.

$$\lim_{h \rightarrow 0} \frac{h^2 - 0 + (-h^2)}{h^2} = 0$$