

Continuity

ISI & CMI Yearlong Course 2024

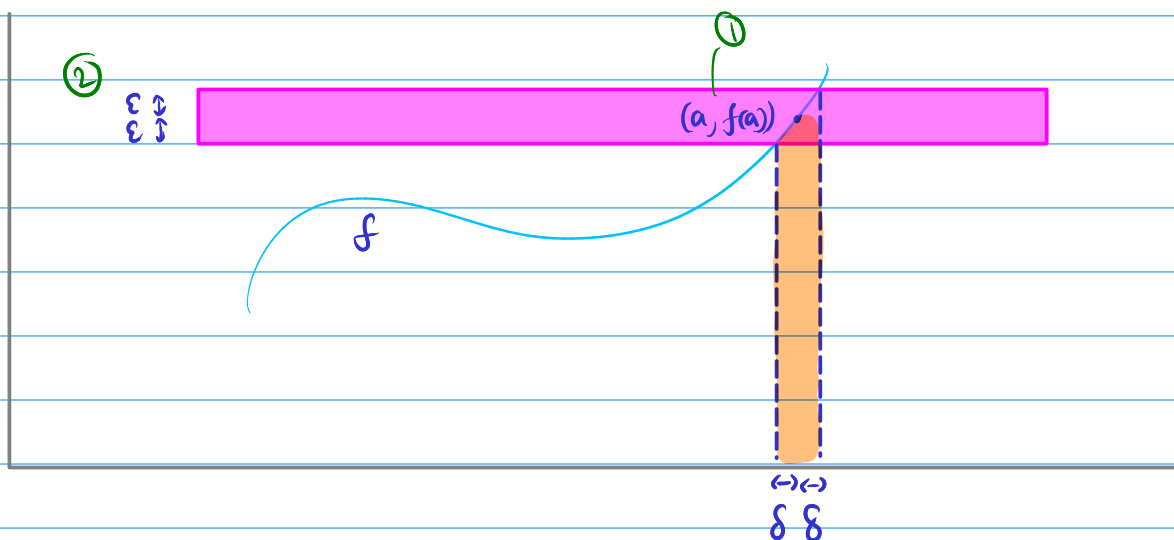
⊙ Introduction

- Defn: Let $f: A \subseteq X \rightarrow Y$. We say f is continuous @ $a \in A$ iff:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

✓ { iff $\forall \varepsilon > 0, \exists \delta > 0 \mid d_X(x, a) < \delta, x \in A \Rightarrow d_Y(f(x), f(a)) < \varepsilon$.

- Alternate Defn: ... is continuous @ $a \in A$ iff $\forall \{x_n\} \subseteq A, x_n \rightarrow a \implies f(x_n) \rightarrow f(a)$



- Result: $f: A \subseteq X \rightarrow Y$ is continuous $\Rightarrow$

f maps neighbourhoods into neighbourhoods.

$$f: B_\delta(a) \xrightarrow{\subseteq A} \bigcirc \in B_\varepsilon(f(a)) \subseteq Y$$

doesn't really matter.

- Result: A $f^n f: X \rightarrow Y$ is continuous iff
 $\forall U \subseteq Y, \underset{\text{open}}{f^{-1}(U)}$ is open in X .

The image of an open set need not be open.
 But if the image set is open, the pre-image set is also!

- Ex: $f(x) = x^2, f: \mathbb{R} \rightarrow \mathbb{R}$.

open $(-2, 2) \mapsto [0, 4)$ not open
 $\mapsto (1, 4)$ open

$\hookrightarrow (-2, -1) \cup (1, 2)$ open ✓

- Result: $f: A \subseteq X \rightarrow Y$, If a is an isolated point of A , then f is continuous @ a .

Proof:

$\because a$ is isolated in $A, \exists \delta_0 > 0 \mid B_{\delta_0}(a) = \{a\}$

Now, $\forall \varepsilon > 0$, take $\delta = \delta_0$.

$\because B_\delta(a) = \{a\}$ then the only point that needs to be checked, is a .

$f(a) \in B_\varepsilon(f(a)). \therefore$ Continuous.

- We say f is continuous on A , if it is continuous @ $a \forall a \in A$.

③ Algebra of Continuous Functions

- If $f, g : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ are continuous @ $a \in A$.
 $\forall \lambda \in \mathbb{R}$, $f + \lambda g$ is continuous @ $a \in A$.
 fg is continuous @ $a \in A$.
- If $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is continuous @ $a \in A$.
 $g : f(A) \rightarrow \mathbb{R}$ is continuous @ $f(a)$.
Then: $g \circ f$ is continuous @ a .

(HW: Prove using definition)

- If $f, g : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ are continuous @ $a \in A$ & g is non-vanishing in a neighbourhood of a , then f/g is continuous.
- The set of all continuous functions on $\mathbb{R}$ written as $C(\mathbb{R})$ forms a commutative ring.
 You can do algebra over this ring & it will be consistent with arithmetic. (not a field, though)

② Some Standard Functions

• Dirichlet / Indicator

$$\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 0 & , \quad x \notin \mathbb{Q} \end{cases}$$

This function is discontinuous everywhere on $\mathbb{R}$
(why?)

• Thomae's Function

$$T(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbb{Q} \setminus \{0\} \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

This function is continuous on $\mathbb{R} \setminus \mathbb{Q}$

(h) Let $A := \{x \in \mathbb{R} : x > 0\}$. For any irrational number $x > 0$ we define $h(x) := 0$. For a rational number in A of the form m/n , with natural numbers m, n having no common factors except 1, we define $h(m/n) := 1/n$. (We also define $h(0) := 1$.)

We claim that h is continuous at every irrational number in A , and is discontinuous at every rational number in A . (This function was introduced in 1875 by K. J. Thomae.)

Indeed, if $a > 0$ is rational, let (x_n) be a sequence of irrational numbers in A that converges to a . Then $\lim(h(x_n)) = 0$, while $h(a) > 0$. Hence h is discontinuous at a .

On the other hand, if b is an irrational number and $\varepsilon > 0$, then (by the Archimedean Property) there is a natural number n_0 such that $1/n_0 < \varepsilon$. There are only a finite number of rationals with denominator less than n_0 in the interval $(b-1, b+1)$. (Why?) Hence $\delta > 0$ can be chosen so small that the neighborhood $(b-\delta, b+\delta)$ contains no rational numbers with denominator less than n_0 . It then follows that for $|x-b| < \delta, x \in A$, we have $|h(x) - h(b)| = |h(x)| \leq 1/n_0 < \varepsilon$. Thus h is continuous at the irrational number b .

Consequently, we deduce that Thomae's function h is continuous precisely at the irrational points in A . (See Figure 5.1.2.) □

2023-10-25

Happy
Dussehra

⑥ EVT & IVT

• Result: Boundedness Theorem.

Let $S \subseteq X$ be compact & $f: S \rightarrow Y$ be continuous.
Then $f(S)$ is bounded.

Proof: (for $X = \mathbb{R}^k$ for some k , $Y = \mathbb{R}$)

Suppose $f(S)$ is NOT bounded.

$\therefore \forall n \in \mathbb{N}, \exists x_n \in S \mid f(x_n) > n$.
Thus we have a sequence: $\{x_n\}$.

$\therefore S$ compact $\Rightarrow S$ is bounded.

$\Rightarrow \{x_n\}$ is a bounded sequence

$\Rightarrow \{x_n\}$ has a convergent subsequence, say $\{x_{n_r}\}$
[Bolzano-Weierstrass]

Say $x_{n_r} \rightarrow x$.

Now, x is a limit point of S , $\therefore \{x_{n_r}\}_{r \in \mathbb{N}} \subseteq S$
 $\therefore S$ closed, $S \ni x$.

Now, $x_{n_r} \rightarrow x$, all $x_{n_r} \in S$. f is continuous on S .
 $\therefore f(x_{n_r}) \rightarrow f(x) \in \mathbb{R}$

However, $\forall x_{n_r}, f(x_{n_r}) > n_r$,
So $f(x_{n_r}) \rightarrow \infty$.

This is a contradiction! $\Rightarrow \Leftarrow$

HW: How to extend to $Y = \mathbb{R}^k$

- Thus we have: for any real-valued continuous function on a compact set, the range is bounded:

$$\exists m, M \in \mathbb{R} \mid \forall x \in S \text{ compact, } m < f(x) < M$$

However, something stronger holds!

- Extremal Value Theorem

$$\begin{aligned} &\forall S \text{ closed \& bounded, } f: S \rightarrow \mathbb{R} \text{ continuous} \\ &\exists m, M \in \mathbb{R} \mid \forall x \in S, m \leq f(x) \leq M. \\ &\& \exists x_{\min}, x_{\max} \in S \mid f(x_{\min}) = m \& f(x_{\max}) = M. \end{aligned}$$

Proof: $M = \sup f(S)$ (very similar for $m = \inf f(S)$)

Now, $\because \mathbb{R}$ has sup property,

$\forall n, M - \frac{1}{n}$ is not an upper bound of $f(S)$

$$\Rightarrow \forall n, \exists x_n \in S \mid M - \frac{1}{n} < f(x_n) \leq M.$$

Now, $f(x_n) \rightarrow M$, also, $\{x_n\}$ is bounded.

By Bolzano-Weierstrass, $\exists \{x_{n_r}\} \mid x_{n_r} \rightarrow x$
 $\because S$ closed, $S \ni x \Rightarrow x_{n_r} \rightarrow x$ in S

By continuity of f : $f(x_{n_r}) \rightarrow f(x)$

$$\text{But, } f(x_{n_r}) \rightarrow M \Rightarrow \underline{\underline{f(x) = M}}$$

$\therefore$ Found $x_{\max} := x$ (similar for $x_{\min}$)

Intermediate Value Property

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to have intermediate value property iff: $\forall x, y \in A, x < y$

$\forall \xi$ between $f(x), f(y)$, $\exists z \in (x, y) \cap A$ |
 $\in (\min\{f(x), f(y)\}, \max\{f(x), f(y)\})$ $f(z) = \xi$

→ Eg: $x \mapsto e^x$ on $(0, \infty)$ ✓ | $x \mapsto 1/x$ on $(-1, 1)$ ✗
 $x \mapsto x^2$ on $\mathbb{Q}$ ✗
 (Id) $x \mapsto x$ on $\mathbb{Q}$ ✗ | $x \mapsto \ln x$ on $(0, 1)$ ✗
 $x \mapsto [x]$ on $\mathbb{R}$ ✗ | $x \mapsto \ln x$ on $(0, 1)$ ✓
 → functions are continuous, but $\mathbb{Q}$ is NOT complete!

$x \mapsto x$ on $[1, 2] \cup [3, 4]$ ✗
 continuity holds, set is complete, but NOT connected!

Intermediate Value Theorem

extension, used often

- equivalent {
- (i) IVP holds for a continuous function on a complete connected set in $\mathbb{R}$.
 - (ii) A continuous function maps a compact, complete, connected set in $\mathbb{R}$ to a compact, complete, connected set in $\mathbb{R}$. *
- (target)

(Fact $\Rightarrow$ compact, complete, connected set = intervals) (closed & bounded)



Theorem [edit]

The intermediate value theorem states the following:

Consider an interval $I = [a, b]$ of real numbers $\mathbb{R}$ and a continuous function $f: I \rightarrow \mathbb{R}$. Then

- Version I. if u is a number between $f(a)$ and $f(b)$, that is,

$$\min(f(a), f(b)) < u < \max(f(a), f(b)),$$

then there is a $c \in (a, b)$ such that $f(c) = u$.

- Version II. the image set $f(I)$ is also an interval (closed), and it contains $[\min(f(a), f(b)), \max(f(a), f(b))]$.

$\Rightarrow$ (implies)



→ Restatement: Let $f: I = [a, b] \rightarrow \mathbb{R}$ be continuous/
 $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$

Then, $\exists c \in (a, b) \mid f(c) = 0$

(This works, as we can always translate by arbi. ξ)

→ Proof: (Bisection & Nested Intervals)

5.3.5 Location of Roots Theorem Let $I = [a, b]$ and let $f: I \rightarrow \mathbb{R}$ be continuous on I . If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.

Proof. We assume that $f(a) < 0 < f(b)$. We will generate a sequence of intervals by successive bisections. Let $I_1 := [a_1, b_1]$, where $a_1 := a, b_1 := b$, and let p_1 be the midpoint $p_1 := \frac{1}{2}(a_1 + b_1)$. If $f(p_1) = 0$, we take $c := p_1$ and we are done. If $f(p_1) \neq 0$, then either $f(p_1) > 0$ or $f(p_1) < 0$. If $f(p_1) > 0$, then we set $a_2 := a_1, b_2 := p_1$, while if $f(p_1) < 0$, then we set $a_2 := p_1, b_2 := b_1$. In either case, we let $I_2 := [a_2, b_2]$; then we have $I_2 \subset I_1$ and $f(a_2) < 0, f(b_2) > 0$.

We continue the bisection process. Suppose that the intervals $I_1, I_2, \dots, I_k$ have been obtained by successive bisection in the same manner. Then we have $f(a_k) < 0$ and $f(b_k) > 0$, and we set $p_k := \frac{1}{2}(a_k + b_k)$. If $f(p_k) = 0$, we take $c := p_k$ and we are done. If $f(p_k) > 0$, we set $a_{k+1} := a_k, b_{k+1} := p_k$, while if $f(p_k) < 0$, we set $a_{k+1} := p_k, b_{k+1} := b_k$. In either case, we let $I_{k+1} := [a_{k+1}, b_{k+1}]$; then $I_{k+1} \subset I_k$ and $f(a_{k+1}) < 0, f(b_{k+1}) > 0$.

If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done. If the process does not terminate, then we obtain a nested sequence of closed bounded intervals $I_n := [a_n, b_n]$ such that for every $n \in \mathbb{N}$ we have

$$f(a_n) < 0 \quad \text{and} \quad f(b_n) > 0.$$

Furthermore, since the intervals are obtained by repeated bisection, the length of I_n is equal to $b_n - a_n = (b - a)/2^{n-1}$. It follows from the (Nested Intervals Property 2.5.2 that there exists a point c that belongs to I_n for all $n \in \mathbb{N}$). Since $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$ and $\lim(b_n - a_n) = 0$, it follows that $\lim(a_n) = c = \lim(b_n)$. Since f is continuous at c , we have

$$\lim(f(a_n)) = f(c) = \lim(f(b_n)).$$

The fact that $f(a_n) < 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim(f(a_n)) \leq 0$. Also, the fact that $f(b_n) > 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim(f(b_n)) \geq 0$. Thus, we conclude that $f(c) = 0$. Consequently, c is a root of f . Q.E.D.

Bartle-Shorbert pp 137 || See also
BS 5.3.5-5.3.10

describing
bisection

2023-10-27

Back to
work...

② Uniform Continuity

- Motivation: What is continuity?
 $x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$

$$\Leftrightarrow \forall x \in A, \forall \varepsilon > 0, \exists \delta > 0 \mid f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

NOTE: δ may depend on x ! (not just on ε)

Consider $f(x) = \sin(\frac{1}{x})$ for $x > 0$
Consider $x = 0.000001 = 10^{-6}$

$f(x)$ is continuous @ $x = 10^{-6}$, but what is δ ...
for a small ε , say 10^{-2} ?

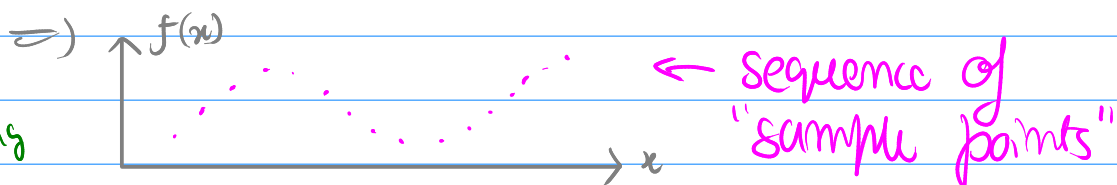
OBSCENELY SMALL: $\sim 10^{-10}$

But @ $x = 1000$, $\varepsilon = 10^{-2}$: $\delta > 10$ works!

$\therefore \sin(\frac{1}{x})$ is continuous $\forall x > 0$,
but near 0, function oscillates too rapidly for
continuity to be practical...

ASIDE: What does functional data look like?

Eg: graph paper.
uniform sampling



If the function oscillates too rapidly (or in some other scenarios)
your sampling breaks down... (BAD)

- Defn: $f : A \subseteq X \rightarrow Y$ is said to be uniformly continuous if:

$$\forall \varepsilon > 0, \exists \delta > 0 \mid \forall x \in A \\ f(B_\delta(x) \cap A) \subseteq B_\varepsilon(f(x))$$

now, the same δ must work $\forall x$! $\rightsquigarrow$ "uniform"

- Eg: Let examine $f(x) = \sin(\frac{1}{x})$ again

Fix any small δ (say $< 10^{-3}$)

$$\text{Fix } x = (1+\delta)\delta = \delta + \delta^2$$

$$\therefore B_\delta(x) = (\delta^2, \delta^2 + 2\delta) \ni \delta^2 + \delta^{100}$$

$$\text{Now, } 1/x : (\delta^2, \delta^2 + 2\delta) \mapsto \left(\frac{1}{\delta^2 + 2\delta}, \frac{1}{\delta^2}\right)$$

$$\text{Width of this interval} = \frac{1}{\delta^2} - \frac{1}{\delta^2 + 2\delta}$$

$$= \frac{1}{\delta} \left(\frac{1}{\delta} - \frac{1}{\delta+2} \right)$$

$$\text{Now } \delta \ll 1 \Rightarrow (2+\delta)^{-1} \approx \frac{1}{2} - \frac{\delta}{4} \approx \frac{1}{2}$$

$$\frac{1}{\delta} > 1000 \quad \therefore \frac{1}{\delta} \gg \frac{1}{\delta+2}$$

$$\therefore \frac{1}{\delta^2} - \frac{1}{\delta^2 + 2\delta} \approx \frac{1}{\delta^2} > 10^6$$

negligible---

$$\gg 2\pi$$

Now, $\sin$ has period 2π ,

So, for any small δ , we cannot achieve $\varepsilon < 1$
 $\Rightarrow$ NOT uniformly continuous

- Important Observation:

It makes sense to talk about continuity at points, but uniform continuity only makes sense for sets...

- Eg: $f(x) = x^2$ on $\mathbb{R}$

Suppose $\exists$ such a δ ...

Fix $x = 2/\delta \Rightarrow B_\delta(x) = (2/\delta \pm \delta)$

Now, $f'(x)$ @ $x = 2/\delta$ is $\rightarrow 4/\delta$.

$$\therefore f\left(\frac{2}{\delta} + \delta\right) > f\left(\frac{2}{\delta}\right) + \delta f'\left(\frac{2}{\delta}\right) \quad \left[\begin{array}{l} \text{if } f \text{ is} \\ \text{convex} \end{array} \right]$$

$$= f\left(\frac{2}{\delta}\right) + 4$$

$\therefore$ Oscillation of f over $(2/\delta \pm \delta)$ is at least 4
(as $2/\delta \in (2/\delta \pm \delta)$)

$\Rightarrow$ Can't make ϵ arbitrarily small
 $\Rightarrow$ NOT uniformly continuous.

- HW: Prove that $\ln(x)$ is uniformly continuous on $[1, \infty)$ but not on $(0, \infty)$

f is said to be convex if
 $\forall \lambda \in (0,1)$
 $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

- Nonuniform Continuity Criteria

TFAB:

i) f is not uniformly continuous on A

ii) $\forall \epsilon > 0, \forall \delta > 0 \exists x, y \in A \mid |x-y| < \delta$
but, $|f(x) - f(y)| \geq \epsilon$

iii) $\exists$ sequences $\{x_n\}, \{y_n\}$ in $A \mid |x_n - y_n| \rightarrow 0$
but $|f(x_n) - f(y_n)| \not\rightarrow 0$

- Lipschitz Criterion for Uniform Continuity

→ A function is said to be Lipschitz ^{on A} if $\exists K > 0 \mid$

$\forall x, y \in A, \quad |f(x) - f(y)| \leq K |x - y|$

--- where K is called the Lipschitz constant.

→ Easy to see: Lipschitz $\Rightarrow$ uniformly continuous.

But, the converse need not hold!

Eg.: $\sqrt{x}$ on $[0, 2]$ (proof will come later)

→ Relaxation of Lipschitz Criterion (extra!!)

You only need to show: $|x-y| < \delta \Rightarrow |f(x) - f(y)| \leq K |x-y|$

Because, if you can find (for a continuous f) $x, y \mid$

$|f(x) - f(y)| > K |x-y|, \exists u, v \mid |u-v| < \delta, |f(u) - f(v)| > K |u-v|$

Equivalence of Continuity & Uniform Continuity on Compact Sets.

$f: A \subseteq X \rightarrow Y$, A is compact.

Then: f is uniformly continuous on A
 $\Leftrightarrow f$ is continuous on A . $\left. \begin{array}{l} \text{"} \Rightarrow \text{"} \\ \text{is trivial} \end{array} \right\}$

BT: f is continuous $\Rightarrow f$ is uniformly continuous.

Proof: (Ref: 4.19, Walter Rudin pp. 91)

Fix any $\epsilon > 0$

$\star_1 \left\{ \begin{array}{l} \because f \text{ is continuous, } \forall x \in A, \exists \delta_x > 0 \mid \\ y \in B_{\delta_x}(x) \Rightarrow f(y) \in B_{\epsilon/2}(f(x)) \end{array} \right.$

Define: $\mathcal{I}_x := B_{\frac{\delta_x}{2}}(x) \Rightarrow \{\mathcal{I}_x \mid x \in A\}$ is
an open cover of A

$\therefore \exists$ a finite subcover, i.e. $\exists x_1, x_2, \dots, x_n$ $\downarrow$
 $\mathcal{I}_{x_1} \cup \mathcal{I}_{x_2} \cup \dots \cup \mathcal{I}_{x_n} \supseteq A$.
 $\because A$ is compact.

Define $\delta^* = \frac{1}{2} \min \{ \delta_{x_1}, \delta_{x_2}, \dots, \delta_{x_n} \}$
 > 0 because of compactness, finite no. of δ 's $\Rightarrow \min > 0$

Claim: This δ^* works for uniform continuity.

TS1: $\forall x, y \in A, |x - y| < \delta^*$

Fix any $x \in A$. $\Rightarrow |f(x) - f(y)| < \varepsilon$

Now, $\because \{ \mathcal{I}_{x_i} \}_{i=1}^m$ is a cover of A ,

$\forall x \in A, \exists x_i \mid x \in \mathcal{I}_{x_i}$

Now, consider any $y \in \mathcal{B}_{\delta^*}(x)$.

We know, $d_X(x, x_i) < \frac{\delta_{x_i}}{2}$ ($\because x \in \mathcal{I}_{x_i}$)

$$d_X(x, y) < \delta^* \leq \frac{\delta_{x_i}}{2}$$

$$\Rightarrow d_X(y, x_i) < \delta_{x_i} \quad (\text{triangle inequality})$$

$$\Rightarrow y \in \mathcal{B}_{\delta_{x_i}}(x_i)$$

Now, $x \in \mathcal{I}_{x_i} = \mathcal{B}_{\frac{\delta_{x_i}}{2}}(x_i) \subseteq \mathcal{B}_{\delta_{x_i}}(x_i)$

using $\star_1$

$\therefore x, y \in \mathcal{B}_{\delta_{x_i}}(x_i) \Rightarrow f(x), f(y) \in \mathcal{B}_{\frac{\varepsilon}{2}}(f(x_i))$

$$\Rightarrow \underline{d_Y(f(x), f(y))} < \varepsilon \quad (\text{by triangle inequality})$$

{ See Bartle-Showalter: 5.4.3, pp. 143 for a simpler proof that assumes $X = Y = \mathbb{R}$

• "Cauchy - Cauchy" Mapping

$$f: A \subseteq X \rightarrow Y \quad (\text{no further assumptions})$$

f is uniformly continuous iff $\{x_n\} \subseteq A$ Cauchy
 $\Rightarrow \{f(x_n)\}$ Cauchy.

Proof: (only if)

Assume $f: A \subseteq X \rightarrow Y$ uniformly continuous.
 Assume $\{x_n\} \subseteq A$ Cauchy.

Fix any $\varepsilon > 0$. $\because f$ uniformly cts, $\exists \delta > 0$ |
 $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Now, $\because \{x_n\}$ Cauchy, $\exists N \in \mathbb{N}$ | $\forall$
 $m, n > N, m, n \in \mathbb{N}$

$$|x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$$

$\therefore \{f(x_n)\}$ is Cauchy.

(if) Assume $\{x_n\} \subseteq A$ Cauchy $\Rightarrow \{f(x_n)\}$ Cauchy

Suppose not, then $\exists \{x_n\}, \{y_n\} \subseteq A$ |
 seq

$$d_X(x_n, y_n) \rightarrow 0 \text{ but } d_Y(f(x_n), f(y_n)) \not\rightarrow 0$$

Construct a sequence $\{z_n\}$ as:

$$x_1, y_1, x_2, y_2, x_3, y_3, \dots$$

Now, assume $\{x_n\}, \{y_n\}$ to be Cauchy.
How { $\because d_X(x_n, y_n) \rightarrow 0$, we have $\{z_n\}$ Cauchy.

Claim: $\{f(z_n)\}$ is NOT Cauchy

If the above claim holds, then we would have a contradiction, as $\{z_n\} \subseteq A$.
Then, the proof will be complete. _{seq}

Suppose the claim DOESN'T hold, i.e.
 $\{f(z_n)\}$ IS Cauchy.

Will show that then: $d_Y(f(x_n), f(y_n)) \rightarrow 0$

$\because \{f(z_n)\}$ Cauchy, $\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid$

$\forall m, n \in \mathbb{N}, m, n > N, d_Y(f(z_m), f(z_n)) < \varepsilon$

Take: $m = 2k-1, n = 2k$

$\hookrightarrow z_m = x_k \hookrightarrow z_n = y_k$

$\therefore \forall \varepsilon > 0, \exists N \in \mathbb{N} \mid \forall k \geq \left\lceil \frac{N}{2} \right\rceil$ just use N

$d_Y(f(x_k), f(y_k)) < \varepsilon$

$\Rightarrow d_Y(f(x_k), f(y_k)) \rightarrow 0$ as $k \rightarrow \infty$.

Thus we have a contradiction

$\Rightarrow$ Claim holds $\Rightarrow$ Contradiction! $\Rightarrow \Leftarrow$

• Continuous Extension Theorem

(for 1D)

5.4.8 Continuous Extension Theorem A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$.

Proof. ($\Leftarrow$) This direction is trivial.

($\Rightarrow$) Suppose f is uniformly continuous on (a, b) . We shall show how to extend f to a ; the argument for b is similar. This is done by showing that $\lim_{x \rightarrow a} f(x) = L$ exists, and this is accomplished by using the sequential criterion for limits. If (x_n) is a sequence in (a, b) with $\lim(x_n) = a$, then it is a Cauchy sequence, and by the preceding theorem, the sequence $(f(x_n))$ is also a Cauchy sequence, and so is convergent by Theorem 3.5.5. Thus the limit $\lim(f(x_n)) = L$ exists. If (u_n) is any other sequence in (a, b) that converges to a , then $\lim(u_n - x_n) = a - a = 0$, so by the uniform continuity of f we have

$$\begin{aligned} \lim(f(u_n)) &= \lim(f(u_n) - f(x_n)) + \lim(f(x_n)) \\ &= 0 + L = L. \end{aligned}$$

Since we get the same value L for every sequence converging to a , we infer from the sequential criterion for limits that f has limit L at a . If we define $f(a) := L$, then f is continuous at a . The same argument applies to b , so we conclude that f has a continuous extension to the interval $[a, b]$. Q.E.D.

Since the limit of $f(x) := \sin(1/x)$ at 0 does not exist, we infer from the Continuous Extension Theorem that the function is not uniformly continuous on $(0, b]$ for any $b > 0$. On the other hand, since $\lim_{x \rightarrow 0} x \sin(1/x) = 0$ exists, the function $g(x) := x \sin(1/x)$ is uniformly continuous on $(0, b]$ for all $b > 0$.

(generalisation)

Assume X, Y complete.

For any bounded A , $f: A \subseteq X \rightarrow Y$,

f is uniformly continuous on A iff it has a continuous extension on $\overline{A}$

↖ "closure" of A .

↳ (not sure about this.)

① Polynomial Approximations to Continuous Functions

Ref: <https://nonagon.org/ExLibris/bernstein-proves-weierstrass>

• What are "approximations"?

Suppose you have an unknown function f . You only know that the function is continuous. How would you estimate the values of f at different points, given the values at some intermediate points?
(also called interpolation)

What does it mean to approximate a function, even?

→ Suppose you have $f: A \subseteq X \rightarrow Y$
we say a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ approximates f if $\forall x \in A$,
 $f_n(x) \rightarrow f(x)$.

(you can also define pointwise Cauchy)

This is also called "pointwise convergence" and is written as: $f_n \rightarrow f$ (pointwise) on A
may be omitted.

→ However, this doesn't always work as we want it to... eg: $f_n = x^n$ on $[0, 1]$
 $f_n \rightarrow 0$ pointwise on $[0, 1]$
 $\therefore \forall x \in [0, 1], x^n \rightarrow 0$.

But we know, $f_n(1) = 1 \forall n$, f_n continuous.

$\therefore$ The convergence doesn't really work near 1.

(see on Desmos) (Recall: uniform continuity... $\Rightarrow$ uniform convergence!!)

→ Pointwise convergence:

$$\forall x \in A, f_n(x) \rightarrow f(x)$$

$$\Leftrightarrow \forall x \in A, \forall \varepsilon > 0, \exists N \in \mathbb{N} \mid \forall n \geq N \mid f_n(x) - f(x) < \varepsilon$$

Uniform convergence:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \mid \forall n \geq N$$

$$\forall x \in A, \mid f_n(x) - f(x) \mid < \varepsilon$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N} \mid \forall n \geq N$$

sup of values
of $\mid f_n(x) - f(x) \mid$
as $x \in A$.

$$\sup_{x \in A} \mid f_n(x) - f(x) \mid < \varepsilon$$

$$L_\infty = \parallel f_n - f \parallel_{\text{sup}}, \parallel f_n - f \parallel_\infty$$

ε may
have
equality?

$$\Leftrightarrow \parallel f_n - f \parallel_{\text{sup}} \rightarrow 0$$

L_∞
norm
over function
spaces

This is also written as:

$$f_n \rightarrow f \text{ uniformly on } A$$

$$\Leftrightarrow f_n \xrightarrow{u} f$$

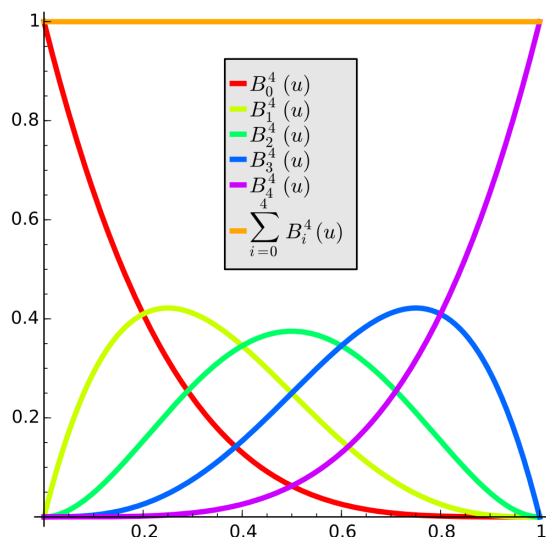
We say $\{f_n\}$ uniformly approximates f .

// We saw pointwise convergence doesn't preserve continuity necessarily; but uniform convergence does.

Bernstein Basis Polynomials

$$\forall n \in \mathbb{N}, \quad k=0, \dots, n \quad B_{k,n}(x) := \binom{n}{k} x^k (1-x)^{n-k}$$

$$B_{k,n} : [0,1] \rightarrow [0,1] \subseteq \mathbb{R}$$



$$n=4$$

$$k=0,1,2,3,4$$

$B_{k,n}^n$ graphs...

[Binomial Distribution]

Verify @ home, use:
 $k \binom{n}{k} = n \binom{n-1}{k-1}$

Facts:

$$\sum_{k=0}^n B_{k,n}(x) = 1, \quad \sum_{k=0}^n k B_{k,n}(x) = nx$$

~ Expectation

$$\sum_{k=0}^n k(k-1) B_{k,n}(x) = n(n-1)x^2$$

Variance

Result:

$$\sum_{k=0}^n (k-nx)^2 B_{k,n}(x) = nx(1-x) \leq \frac{n}{4} \quad \forall x \in [0,1]$$

$$\Rightarrow \sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 B_{k,n}(x) = \frac{x(1-x)}{n} \leq \frac{1}{4n} \quad \forall x \dots$$

Bernstein Polynomials for a Continuous Function

$$f_n(x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x)$$

degree n polynomial

$$= \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

n = no. of trials
 x = prob. of success
 Z = no. of success (random)

expectation! $f_n(x) = \mathbb{E}\left[f\left(\frac{Z}{n}\right)\right]$

reword for $[Z=k]$

$P(Z=k)$ where $Z \sim \text{Binomial}(n, x)$

- Weierstrass Approximation Theorem

For every continuous function $f: [0, 1] \rightarrow \mathbb{R}$,
 $\exists$ a sequence of polynomials that uniformly approximates it.

Shall show: $f_n \xrightarrow{u} f$ on $[0, 1]$

Let $x \in [0, 1]$ and $\delta > 0$

$$\begin{aligned} \sum_{k: |\frac{k}{n} - x| \geq \delta} B_{k,n}(x) &\leq \frac{1}{\delta^2} \sum_{k: |\frac{k}{n} - x| \geq \delta} \left(\frac{k}{n} - x\right)^2 B_{k,n}(x) \\ &\leq \frac{1}{\delta^2} \sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 B_{k,n}(x) \\ &\leq \frac{1}{\delta^2} \frac{1}{4n} = \frac{1}{4\delta^2 n} \end{aligned}$$

Now $\left| f(x) \cdot 1 - \sum_{k: |\frac{k}{n} - x| \geq \delta} f\left(\frac{k}{n}\right) B_{k,n}(x) \right|$

$$\begin{aligned} &= \left| f(x) \sum_k B_{k,n}(x) - \sum_{k: \dots} f\left(\frac{k}{n}\right) B_{k,n}(x) \right| \\ &\leq \sum_{k: \dots} \left| f(x) - f\left(\frac{k}{n}\right) \right| B_{k,n}(x) \end{aligned}$$

$\therefore f$ continuous on $[0, 1]$ (compact), f bounded:

$$\begin{aligned} |f| &< M \quad \forall M \in [0, 1] \\ \Rightarrow |f(x) - f\left(\frac{k}{n}\right)| &< 2M \quad \forall \dots \end{aligned}$$

$$< 2M \sum_{k: \dots} B_{k,n}(x)$$

$$\leq 2M \cdot \frac{1}{4\delta^2 n} = \frac{M}{2\delta^2 n}$$

Now, fix $\varepsilon > 0$. By uniform continuity of f

$$\exists \delta > 0 \mid |u-v| < \delta \Rightarrow |f(u) - f(v)| < \frac{\varepsilon}{2}$$

$$\therefore \left| f(x) - \sum_{k: \left| \frac{k}{n} - x \right| < \delta} f\left(\frac{k}{n}\right) B_{k,n}(x) \right|$$

$$\leq \sum_{k: \dots} \underbrace{|f(x) - f\left(\frac{k}{n}\right)|}_{< \varepsilon/2} B_{k,n}(x) < \varepsilon/2. \quad \textcircled{1}$$

Now, for this δ .

$$\left| f(x) - \sum_{k: \left| \frac{k}{n} - x \right| \geq \delta} f\left(\frac{k}{n}\right) B_{k,n}(x) \right| \leq \frac{M}{2\delta^2 n}$$

$$\text{Choose } n \text{ large} \mid \frac{M}{2\delta^2 n} < \varepsilon/2. \quad \textcircled{2}$$

same sum, no restriction.

$$\therefore |f(x) - f_n(x)| < \underbrace{\varepsilon/2 + \varepsilon/2}_{\text{By } \textcircled{1} \& \textcircled{2}} = \varepsilon \quad \forall x \in [0,1]$$

$$\Rightarrow f_n \xrightarrow{u} f.$$