

2023-11-13
Happy
Diwali!

Differentiability & Derivatives

ISI & CMI Yearlong Course 2024

③ Introduction

- Defn: Let $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $\forall x \in U$, if $\exists \ell \in \mathbb{R}$ | U open

$$\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = \ell$$

--- Then, f is said to be differentiable @ x , with derivative ℓ , written as $f'(x) = \ell$.

- Example: $f(x) = \sin(x)$ @ any $x \in \mathbb{R}$.

$$\lim_{y \rightarrow x} \frac{\sin y - \sin x}{y - x} = \lim_{y \rightarrow x} \frac{\cancel{2} \sin\left(\frac{y-x}{2}\right) \cos\left(\frac{y+x}{2}\right)}{\cancel{2} \cdot \left(\frac{y-x}{2}\right)}$$

$$= \lim_{y \rightarrow x} \frac{\sin\left(\frac{y-x}{2}\right)}{\left(\frac{y-x}{2}\right)} \lim_{y \rightarrow x} \cos\left(\frac{y+x}{2}\right)$$

$\underbrace{\left(\frac{y-x}{2}\right)}_{=: h}$

$$= \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \cos(x) \quad \left[\text{By continuity of } \cos x \right]$$
$$= \cos(x)$$

- Result: $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ differentiable @ $x \in U$ $\Rightarrow f$ continuous @ $x \in U$.

Proof: $\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = \ell \in \mathbb{R}$, $\Delta (y - x) \rightarrow 0$
 $\therefore$ Multiplying, $f(y) - f(x) \rightarrow 0 \Rightarrow f(y) \rightarrow f(x)$

⊙ Left-Hand & Right-Hand Derivatives

- $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be left-differentiable @ $x \in U \subseteq \mathbb{R}$ if $\exists l \in \mathbb{R}$
 $x \in U \subseteq \mathbb{R}$ _{open} $[-\infty, x]$

$$\lim_{y \rightarrow x^-} \frac{f(y) - f(x)}{y - x} = l$$

where l is called the left-hand / left derivative @ x .

$$\equiv \lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} = l$$

$$\equiv \lim_{h \downarrow 0} \frac{f(x-h) - f(x)}{-h} = l$$

} all are equivalent.

- (similar for right-hand / right derivative)

$$\lim_{y \rightarrow x^+} \frac{f(y) - f(x)}{y - x} = \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \downarrow 0} \dots = l$$

• Results:

- i) left differentiable $\Rightarrow$ left continuous
right differentiable $\Rightarrow$ right continuous.
- ii) Both exist & are equal $\Rightarrow$ differentiable
- iii) Both exist but not equal $\Rightarrow$ continuous but not diff?
 (try 121)

• Warning

$$\lim_{h \downarrow 0} \frac{f(x+h) - f(x-h)}{2h} = l \in \mathbb{R} \not\Rightarrow \text{differentiable}$$

$$f(x) = \begin{cases} x^3, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Continuity @ x is not guaranteed.

⊙ Classes of Functions based on Differentiability

- $\forall f$ differentiable, define $Df : x \mapsto f'(x)$
on $U \subseteq \mathbb{R}$ $U \rightarrow \mathbb{R}$

Write: $f^{(0)} \equiv f$, $f^{(r)} \equiv Df^{(r-1)} \quad \forall r=1, \dots, N$

--- where $N \mid f^{(N)}$ not differentiable on U .

- Define $C^r(U)$; $U \subseteq \mathbb{R}$ open:

$$C^r(U) = \{f \mid f^{(r)} \text{ exists \& is continuous on } U\}$$

- Notice $\because$ differentiability $\Rightarrow$ continuity.
we have $C^0(U) \supseteq C^1(U) \supseteq C^2(U) \supseteq \dots \supseteq C^k(U) \supseteq \dots$
 $\therefore C^k(U) \downarrow C^\infty(U)$ as $k \rightarrow \infty$.

$C^\infty(U) \Rightarrow$ define (algebra) of all infinitely differentiable fns.
read "set" on U .

- Observation: All polynomials are C^∞ on $\mathbb{R}$.

- Results: $f, g : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$

$$\left. \begin{aligned} D(f + \lambda g) &= (Df) + \lambda(Dg) \\ D(fg) &= f(Dg) + (Df)g \end{aligned} \right\} \begin{array}{l} \text{addition} \\ \text{multiplication} \\ \text{scaling} \\ \Rightarrow \text{algebra} \end{array}$$

If g is non-vanishing on U :

$$D\left(\frac{f}{g}\right) = \frac{g(Df) - f(Dg)}{g^2}$$

[This page is NOT in syllabus].

- Considering $C^k(U)$ are algebras, what exactly is " $\mathcal{D}$ "?

$$\mathcal{D} : \begin{array}{ccc} f & \mapsto & \mathcal{D}f \\ C^k(U) & \rightarrow & C^{k-1}(U) \end{array}$$

Now, $\mathcal{D}$ is linear as $\mathcal{D}(f+\lambda g) = \mathcal{D}f + \lambda(\mathcal{D}g)$
 $\therefore \mathcal{D}$ is a linear map on $C^k(U)$ to $C^{k-1}(U)$
 $\Rightarrow \mathcal{D} \in \mathcal{L}(C^k(U), C^{k-1}(U))$

Now consider $\mathcal{D}_2$ | $\mathcal{D}_2 : f \mapsto \mathcal{D}(\mathcal{D}f)$
 $C^k(U) \rightarrow C^{k-2}(U)$

We know second derivatives are also linear.

$$\therefore \mathcal{D}_2 \in \mathcal{L}(C^k(U), C^{k-2}(U))$$

What is the relation between $\mathcal{D}$ & $\mathcal{D}_2$?

$$\begin{array}{ccc} C^k(U) \ni f & & \mathcal{D}_2 \cong \mathcal{D} \cdot \mathcal{D} \\ \mathcal{D} \downarrow & \searrow \mathcal{D}_2 & \quad \quad \quad \begin{array}{c} ?? \\ \downarrow \\ \text{Composition} \end{array} \\ C^{k-1}(U) & \xrightarrow{\mathcal{D}} & C^{k-2}(U) \end{array} \quad \quad \quad = \mathcal{D}^2$$

Define: $\mathcal{D}^k(f) = \mathcal{D}(\mathcal{D}^{k-1}(f))$

Multiplication in $\mathcal{L}(C^k(U), \dots)$ turns into composition upon application --- $\rightarrow$ Matrices?!

$\Rightarrow$ Multivariate derivative as LTs / Jacobians.

⊙ Standard Results

- Chain Rule: $D(f \circ g) = (Df)(g) \cdot (Dg)$

- Differentiating inverses:

$$f(f^{-1}(y)) = y \Rightarrow (Df)(f^{-1}(y)) \cdot (Df^{-1})(y) = 1$$

$$\Rightarrow (Df^{-1})(y) = [f'(f^{-1}(y))]^{-1}$$

- (Standard derivatives)

⊙ Curious Case of $x^p \sin(\frac{1}{x}) =: f_p(x)$, $x > 0$, $f_p(x) = 0$ @ $x=0$

- We already showed for $p > 0$, $f_p(x)$ is continuous.

Formally, $\frac{d}{dx} \left(x^p \sin(\frac{1}{x}) \right)$

$$= x^p \cos(\frac{1}{x}) \cdot \frac{-1}{x^2} + p x^{p-1} \sin(\frac{1}{x})$$

$$= p x^{p-1} \sin(\frac{1}{x}) + x^{p-2} \cos(\frac{1}{x})$$

Assume $p > 0$, we want f_p differentiable @ 0.

$$\lim_{h \rightarrow 0} \frac{h^p \sin(\frac{1}{h})}{h} = \lim_{h \rightarrow 0} \underbrace{h^{p-1} \sin(\frac{1}{h})}_{\text{oscillating!}} = 0$$

$\updownarrow$
 $p > 1$

Check for $p > 1$,

Df_p is continuous @ 0. $\rightarrow$ Check for differentiability of Df_p ---

- Result: $x^p \sin(\frac{1}{x})$ is $C^k \Leftrightarrow p > k, \forall k \in \mathbb{N}$.

@ $p = k$, the formal k^{th} derivative has a $x^0 \sin(\frac{1}{x})$ term

• Differentiability & Graph Sketching

Smoothness $\Rightarrow C^1$ (or C^2) $\rightarrow$ "smooth turns"

Sharp corner $\Rightarrow$ non-differentiable

Maxima/minima/inflection point $\Leftarrow$ derivative 0 (if exists)

- Second derivative $> 0 \Leftrightarrow$ minima

- " " " " $< 0 \Leftrightarrow$ maxima

- Inflection points are weird, can have all derivatives 0, even.

2023-11-15 =
IND vs NZ

• Mean Value Theorems

- Fermat's Theorem: $f: (a, b) \rightarrow \mathbb{R}$ has a local extremum at $x_0 \in (a, b)$. If f is differentiable @ x_0 , $f'(x_0) = 0$.

Suppose that x_0 is a local maximum (a similar proof applies if x_0 is a local minimum). Then there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and such that we have $f(x_0) \geq f(x)$ for all x with $|x - x_0| < \delta$. Hence for any $h \in (0, \delta)$ we have

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

Since the limit of this ratio as h gets close to 0 from above exists and is equal to $f'(x_0)$ we conclude that $f'(x_0) \leq 0$. On the other hand, for $h \in (-\delta, 0)$ we notice that

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

but again the limit as h gets close to 0 from below exists and is equal to $f'(x_0)$ so we also have $f'(x_0) \geq 0$.

Hence we conclude that $f'(x_0) = 0$.

- Rolle's Theorem: $f: [a, b] \rightarrow \mathbb{R}$ continuous, f is differentiable on (a, b) . $f(a) = f(b)$
Then $\exists x_0 \in (a, b) \mid f'(x_0) = 0$.

Proof

Case i: f is a constant function. (Trivial, choose any x_0)

Case ii: f is non-constant

$\therefore f$ is continuous on a compact set, attains an absolute maximum and minimum.

f is non-constant, at least one of them is attained in (a, b) , not at endpoints. (else $\max = \min \Rightarrow f_{\text{const}}$)

$\hookrightarrow$ say this happens @ $x_0 \in (a, b)$. ^{for absolute maximum (wlog)} Fix any $\delta > 0$.

Now, $\forall x \in [a, b] \supseteq (x_0 - \delta, x_0 + \delta) \quad f(x) \leq f(x_0)$

$\Rightarrow x_0$ is a local maximum of f .

By Fermat's Theorem, $f'(x_0) = 0$

- Cauchy's Mean Value Theorem

$f, g: [a, b] \rightarrow \mathbb{R}$ continuous ^{on $[a, b]$} & differentiable on (a, b)
Then $\exists x_0 \in (a, b) \mid$

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad , \text{ assuming } g(b) \neq g(a)$$

Proof: Apply Rolle's Theorem to

(JW)

$$h(x) := f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$$

- Lagrange's Mean Value Theorem

$f: [a, b] \rightarrow \mathbb{R}$ continuous, f differentiable on (a, b)
Then $\exists x_0 \in (a, b)$ |

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

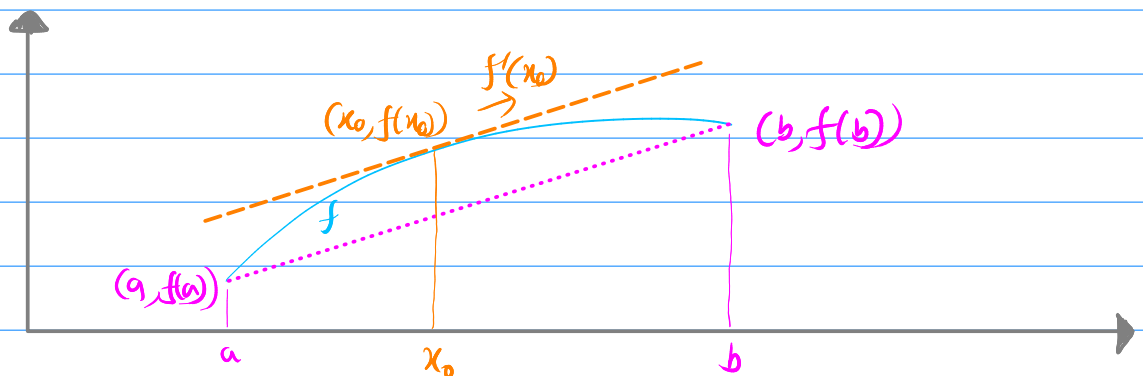
→ Proof: (special case of CMVT w/ $g: x \mapsto x$).

→ Geometric Interpretation

Now $\frac{f(b) - f(a)}{b - a}$ ← slope of secant between $(a, f(a))$ & $(b, f(b))$

& $f'(x)$ denotes slope of tangent to f @ $(x, f(x))$

$\therefore$ LMVT essentially says $\exists$ a tangent $\parallel$ to the secant joining endpoints.



(Recap & details on MVT)

■ Cauchy's Theorem

Let $f : I = [a, b] \rightarrow \mathbb{R}$, $I \ni c$.

f is differentiable @ c , iff $\exists$ a $f^n \phi$ cts @ c |

$$f(x) - f(c) = \phi(x) \cdot (x - c)$$

$$\& \phi(c) = f'(c)$$

→ Proof (assume f differentiable @ c)

$$\text{Define } \phi(x) = \begin{cases} \frac{f(x) - f(c)}{x - c} & , x \neq c \\ f'(c) & , x = c \end{cases}$$

TST: ϕ continuous @ c .

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c) \quad (\text{by defn})$$

→ Proof (assume ϕ exists, cts @ c):

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c} \phi(x)$$

$$= \phi(c) \quad (\text{by continuity})$$

$$\therefore \text{Limit exists} \Rightarrow f'(c) := \phi(c)$$

→ Note: This theorem is used to prove chain rule

• Intermediate Value Property of Derivatives

$f: I = [a, b] \rightarrow \mathbb{R}$ differentiable on I .

Then, f' has IVP on I , even if it is not continuous!

- Lemma Let $f: I = [a, b] \rightarrow \mathbb{R}$, f differentiable @ $c \in I$.

$$a) \quad f'(c) > 0 \Rightarrow \exists \delta > 0 \mid \forall x \in (c, c + \delta) \quad f(x) > f(c)$$

$$b) \quad f'(c) < 0 \Rightarrow \exists \delta > 0 \mid \forall x \in (c - \delta, c) \quad f(x) > f(c)$$

(Proof is HW) $\rightarrow$ Check Bartle-Shurbutt.

• Darboux's Theorem

$f: I = [a, b] \rightarrow \mathbb{R}$ differentiable on I .

Then, if $k \in (f'(a) \wedge f'(b), f'(a) \vee f'(b))$
then $\exists c \in (a, b) \mid f'(c) = k$.

Essentially f' has IVP.

Proof: Suppose $f'(a) < k < f'(b)$.

$$g'(x) := kx - f(x) \leftarrow \text{continuous, hence attains maxima. on } [a, b]$$

$$\Rightarrow g'(x) = k - f'(x)$$

$$\begin{aligned} \rightarrow g'(a) &= k - f'(a) > 0 \Rightarrow \text{no maxima of } g \text{ @ } a \\ \rightarrow g'(b) &= k - f'(b) < 0 \Rightarrow \text{no maxima of } g \text{ @ } b \end{aligned}$$

$\Rightarrow \exists c \in (a, b)$ where g attains maxima (by above lemma)

$$g'(c) = k - f'(c) = 0 \Rightarrow f'(c) = k$$

⑥ L'Hôpital's Rules

- If a limit is of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, where the limit in question is:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$
 then given both f & g are differentiable in an nbd around a , $(a-\delta, a+\delta)$ ✓
 &
$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \in \overline{\mathbb{R}} = [-\infty, \infty]$$
 (includes $\pm\infty$)
 ↳ $\mathbb{R}$ "closure"

Then: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$ || Result holds analogously for $x \rightarrow a^+$, a^- etc.

- Proof: makes use of CMVT on $(a \pm \delta)$, easier for $0/0$ case, use CMVT and send $\delta \rightarrow 0$.

⑦ (Bartle-Shorbut Exc for §6.3)

6. Evaluate the following limits.

(a) $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$ 2

(b) $\lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^4}$ 1/3

7. Evaluate the following limits, where the domain of the quotient is as indicated.

(a) $\lim_{x \rightarrow 0^+} \frac{\ln(x+1)}{\sin x}$ $(0, \pi/2)$, 1

(b) $\lim_{x \rightarrow 0^+} \frac{\tan x}{x}$ $(0, \pi/2)$, 1

(c) $\lim_{x \rightarrow 0^+} \frac{\ln \cos x}{x}$ $(0, \pi/2)$, 0

(d) $\lim_{x \rightarrow 0^+} \frac{\tan x - x}{x^3}$ $(0, \pi/2)$, 1/3

8. Evaluate the following limits:

(a) $\lim_{x \rightarrow 0} \frac{\text{Arctan } x}{x}$ $(-\infty, \infty)$, 1

(b) $\lim_{x \rightarrow 0} \frac{1}{x(\ln x)^2}$ $(0, 1)$, ∞

(c) $\lim_{x \rightarrow 0^+} x^3 \ln x$ $(0, \infty)$, 0

(d) $\lim_{x \rightarrow \infty} \frac{x^3}{e^3}$ $(0, \infty)$, ∞

Try Q12, Q14.

⑨

12. Let f be differentiable on $(0, \infty)$ and suppose that $\lim_{x \rightarrow \infty} (f(x) + f'(x)) = L$. Show that $\lim_{x \rightarrow \infty} f(x) = L$ and $\lim_{x \rightarrow \infty} f'(x) = 0$. [Hint: $f(x) = e^x f(x)/e^x$.]
13. Try to use L'Hospital's Rule to find the limit of $\frac{\tan x}{\sec x}$ as $x \rightarrow (\pi/2)^-$. Then evaluate directly by changing to sines and cosines.
14. Show that if $c > 0$, then $\lim_{x \rightarrow c} \frac{x^c - c^x}{x^x - c^c} = \frac{1 - \ln c}{1 + \ln c}$.

$$x^x = e^{x \ln x}$$

$$\textcircled{12} \quad \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{e^x f(x)}{e^x} = \lim_{x \rightarrow \infty} \frac{e^x (f(x) + f'(x))}{e^x} \quad \left[\begin{array}{l} \text{L'Hô.} \\ \text{Rule} \end{array} \right]$$

New subtract $\lim_{x \rightarrow \infty} f(x) = L \quad \leftarrow \quad = L \quad (\text{given})$

$$\textcircled{14} \quad \lim_{x \rightarrow c} \frac{x^c - c^x}{x^x - c^c} = \lim_{x \rightarrow c} \frac{cx^{c-1} - c^x \ln c}{x^x (1 + \ln x)} = \frac{c^c (1 - \ln c)}{c^c (1 + \ln c)}$$

⑩ Taylor's Theorem & Taylor Series

- Suppose $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is C^∞ (or at least C^n).
Let $x_0 \in U$. open We define:-

$$P_n: x \in U \mapsto \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \quad \leftarrow \begin{array}{l} n^{\text{th}} \text{ order} \\ \text{Taylor} \\ \text{polynomial} \\ \text{of } f \text{ about} \\ x_0. \end{array}$$

- Observation: (check yourself!)

$$\begin{array}{l} f \equiv f^{(0)}, f' \equiv f^{(1)}, f'' \equiv f^{(2)}, \dots, f^{(n)} \\ P_n \equiv P_n^{(0)}, P_n' \equiv P_n^{(1)}, P_n'' \equiv P_n^{(2)}, \dots, P_n^{(n)} \end{array} \quad \begin{array}{l} \text{agree with} \\ \text{on a neighbourhood of} \\ x_0 \in U. \end{array}$$

Hint: Put x_0 in all of the above & pair up!
(continuity takes care of the neighbourhood part)

- Define: $R_n := f - P_n$ $\leftarrow$ n^{th} order remainder of f about $x_0 \in U$.

- Taylor's Theorem. $f : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ C^n
open

Let P_n denote the n^{th} order Taylor polynomial of f about $x_0 \in U$. $R_n := f - P_n$.

If f is $(n+1)$ -times differentiable (not necessarily C^{n+1})
Then $\forall x \in U, x \neq x_0, \exists \xi \in (x \wedge x_0, x \vee x_0)$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}.$$

$\nearrow$ Taylor's form of remainder
 $\sim n=0$ case.

This is a generalisation on LMVT and can be proven using LMVT & induction on 'n'.

- Under some assumptions...

For every C^∞ $f: U \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $P_n \xrightarrow[n \rightarrow \infty]{} 0$
open

This just means $\{P_n\}$ approximates f !

(Under similar assumptions, $\{P_n\}$ uniformly approximates f)

- Hence, the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$ is called
... the Taylor series expansion of f about $x_0 \in U$.

- NOTE: $x_0 = 0$ is also called the Maclaurin Series of f .

- Functions whose Taylor series converges (to the function) and hence, which can be written down as a power series, are called (real) analytic.

The set of all (real) analytic fns over a set A is denoted by $C^\omega(A)$. Crucially, observe that:
 $C^\omega(A) \subseteq C^\infty(A)$.

- Result: Non-analytic smooth fns exist.

$$\Leftrightarrow \exists f \in C^\infty(A) \mid f \notin C^\omega(A).$$

Example: $f(x) = e^{-1/x} \mathbb{1}(x > 0)$ is C^∞

$$f(0) = f'(0) = f^{(2)}(0) = \dots = f^{(n)}(0) = \dots = 0$$

$\Rightarrow$ Taylor series $\equiv 0 \Rightarrow f \equiv 0$ if f were analytic!
 (around 0)



https://en.wikipedia.org/wiki/Non-analytic_smooth_function

✓ (this note is print friendly ☺)

2023-11-23 =

⑥

16. Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \rightarrow \mathbb{R}$ be differentiable on I , and suppose $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

$$\text{Hint: } \lim_{h \rightarrow 0} \frac{\frac{f(a+h) - f(a)}{h} - \frac{f(a) - f(a-h)}{h}}{h}$$

↓
 using
 L'Hôpital
 easy

We know

$$RHS = \lim_{h \rightarrow 0} \frac{\frac{f(a+h) - f(a)}{h} - \frac{f(a) - f(a-h)}{h}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f'(d) - f'(c)}{h} \quad \text{[LMVT] where: } \begin{matrix} d \in (a, a+h) \\ c \in (a-h, a) \end{matrix}$$

Notice that f' need not be continuous @ a for this limit to exist!

$$= \lim_{h \rightarrow 0} \frac{f'(a+\varepsilon) - f'(a-\delta)}{h} \quad \text{where } \varepsilon, \delta \in (0, h)$$

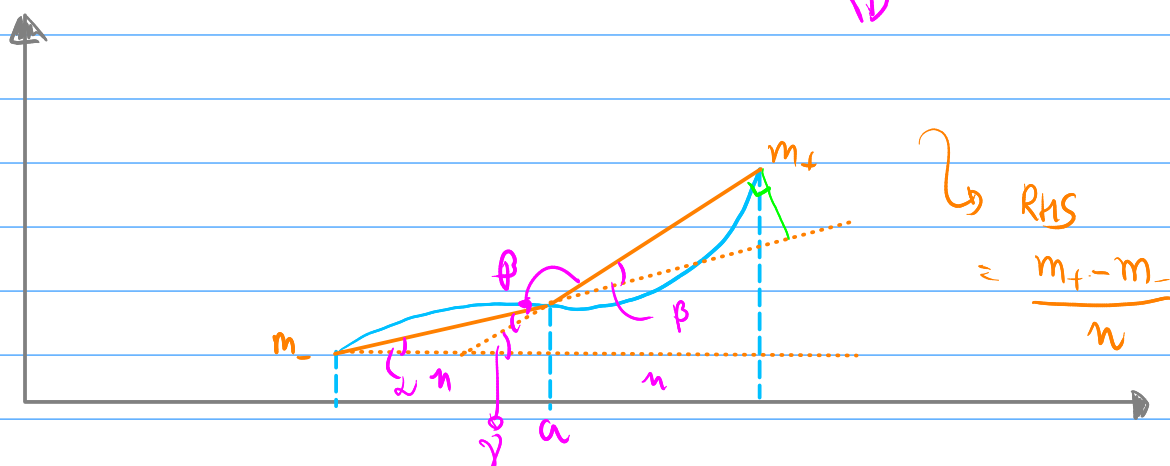
$$= \lim_{h \rightarrow 0} \frac{f'(a+\varepsilon) - f'(a)}{h} + \lim_{h \rightarrow 0} \frac{f'(a) - f'(a-\delta)}{h}$$

$$(?) = \lim_{h \rightarrow 0} (\varepsilon/h + \delta/h) f'(a)$$

$$= f''(a) \lim_{h \rightarrow 0} \left(\frac{d-c}{n} \right) \quad \downarrow$$

there are f's where RHS exists, but f' is discontinuous @ a .

$f''(a) \otimes$



$$m_+ - m_- = \tan \gamma - \tan \alpha = \frac{\sin(\gamma - \alpha)}{\cos(\alpha) \cos(\gamma)}$$

(finish @ home)

$$= \frac{\sin(\beta)}{\cos(\alpha) \cos(\gamma)}$$

② (Fixed-point Iteration Theorem)

22. Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t , prove that f has at most one fixed point. *(IVP of derivatives; either f decreases, or increases faster or slower than x)*

(b) Show that the function f defined by

verify! $f(t) = t + (1 + e^t)^{-1}$

$\Rightarrow \because (1 + e^{-t})^{-1} \neq 0 \forall t$

has no fixed point, although $0 < f'(t) < 1$ for all real t .

★ (c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t , prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

limit: contractive sequence; & Lipschitz function

for $n = 1, 2, 3, \dots$

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \rightarrow (x_2, x_2) \rightarrow (x_2, x_3) \rightarrow (x_3, x_3) \rightarrow (x_3, x_4) \rightarrow \dots$$

$$\because |f'(t)| \leq A \quad \forall t, \quad \forall x, y \in \mathbb{R} \quad |f(x) - f(y)| \leq A|x - y|$$

Why? Otherwise by LMVT

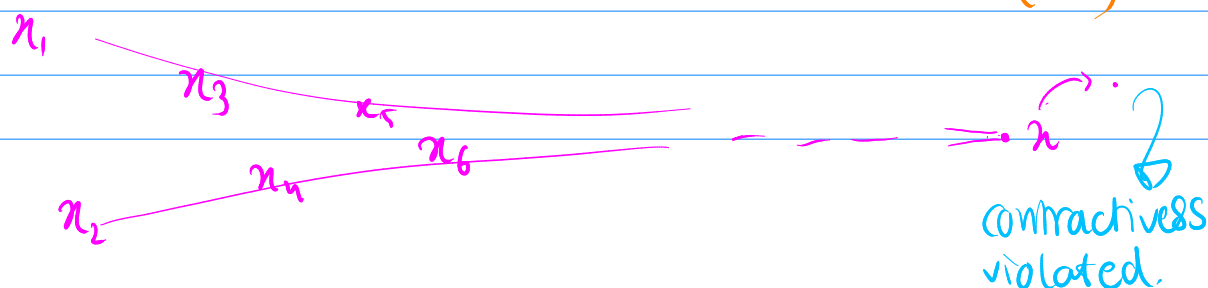
$$\exists z \in (x, y) \mid f'(z) > A$$

$\therefore f$ is Lipschitz with Lipschitz constant $A < 1$

$\Rightarrow \forall x_1, \{x_{n+1} = f(x_n)\}$ is contractive, hence

(due to completeness of $\mathbb{R}$) Cauchy, hence convergent.

Let $x_n \rightarrow x$, suffice to show x is a fixed point. *(HW)*



• (Newton-Raphson Method)

6.4.7 Newton's Method Let $I := [a, b]$ and let $f : I \rightarrow \mathbb{R}$ be twice differentiable on I . Suppose that $f(a)f(b) < 0$ and that there are constants m, M such that $|f'(x)| \geq m > 0$ and $|f''(x)| \leq M$ for $x \in I$ and let $K := M/2m$. Then there exists a subinterval I^* containing a zero r of f such that for any $x_1 \in I^*$ the sequence (x_n) defined by

$$(5) \quad x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for all } n \in \mathbb{N},$$

belongs to I^* and (x_n) converges to r . Moreover

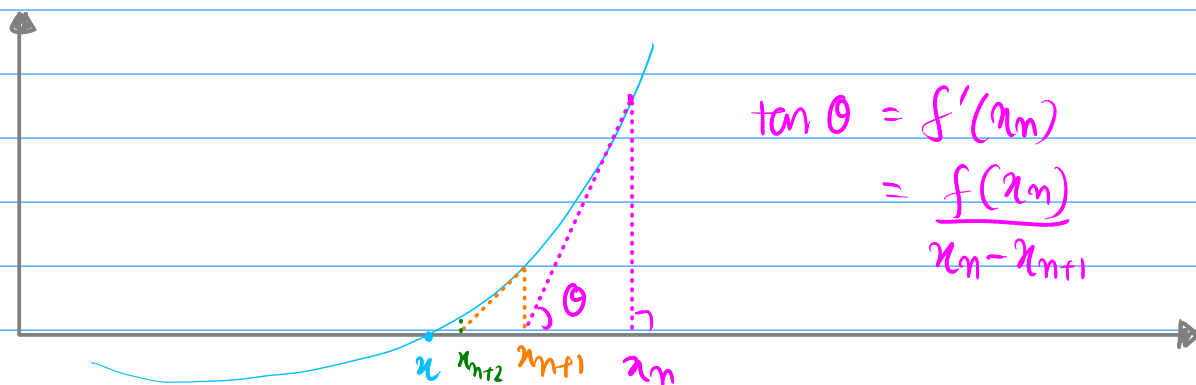
$$(6) \quad |x_{n+1} - r| \leq K |x_n - r|^2 \quad \text{for all } n \in \mathbb{N}.$$

Essentially, let f be a twice differentiable f^n on $[a, b]$.
Assumptions: i) f has a root in I
 ii) $f' \geq m > 0$, i.e. strictly increasing f
 $\Rightarrow f$ has exactly one root in I .

Then, pick any $x_1 \in I$, and construct $\{x_n\}$ as follows:

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

Then, $x_n \rightarrow x$, x is the root of f in I .



$$\therefore f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}} \Leftrightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$