

Calculus - I

ISI & CMI Yearlong Course 2024

⑤ The Real Numbers

⊙ The Rationals

- Defn: $\left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0, \gcd(p, q) = 1 \right\}$
 $=: \mathbb{Q}$ (the set of rational numbers)

- We know how to do arithmetic on the rationals.

- Try to solve $x^2 - 2 = 0$ in $\mathbb{Q}$.
 $\Rightarrow$ Concl (existence of irrational numbers)

- Fact: $\mathbb{Q}$ is a totally ordered field.

✓ Know what a "field" is
✓ Know what a "total order" is.
+ They "agree".

(See Also: Wikipedia on "Ordered Field")

⊙ The Need for a Larger Number System

- We usually only move to a "larger" number system, when it gives us some advantage.

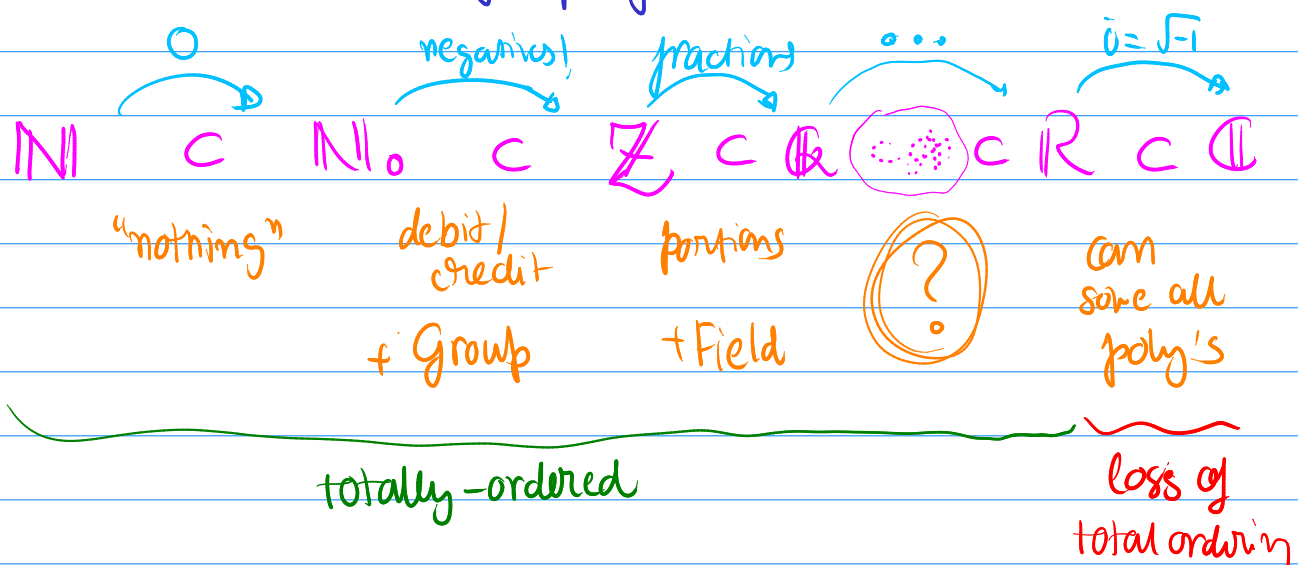
$\Rightarrow$ Can solve polynomials in $\mathbb{R}$! $\otimes$ $x^2 + 1$

So what advantage do we gain?

- We already have: $\checkmark$ Field $\checkmark$ Total Ordering.

We don't have solutions to (many) polynomials but $\mathbb{R}$ doesn't as well!

So, what is this "jump" from $\mathbb{Q} \rightarrow \mathbb{R}$



In other words, what does $\mathbb{R}$ have that $\mathbb{Q}$ doesn't? (in terms of mathematical properties)

Ans: $\downarrow$

- Recall, lub property ($\equiv$ glb property).

$\Rightarrow$ A set is said to have lub property if every subset has a lub (supremum)

Try to find $\sup \{x \mid x^2 < 2\}$ in $\mathbb{Q}$

$\Rightarrow$ Can't.

Provide a proof that you can't!

{ This proof shows that $\mathbb{Q} \subseteq \mathbb{R}$ doesn't have sup property.
But, you need to construct $\mathbb{R}$ axiomatically first. }

Proof: $A = \{x \mid x^2 < 2\}$.

$$A^2 = \{x^2 \mid x \in A\} = \{x \mid x^2 < 2\} \\ = \{y \mid y < 2\} \subseteq \mathbb{Q}$$

Suppose $x_0 = \sup A \in \mathbb{Q}$

$$\forall x \in A, \quad x^2 < 2 \\ x \leq x_0 \Rightarrow x^2 \leq x_0^2$$

Case - I: $x_0^2 = 2$ *Impossible.*

Lemma: $\exists$ a rational no. & irrational no. b/w

- i) any 2 rationals
- ii) any 2 irrationals
- iii) any rational & irrational.

Case - II: $x_0^2 > 2$.

Then choose $x' \in (\sqrt{2}, x_0) \cap \mathbb{Q}$

$$x^2 < 2 < (x')^2 < (x_0)^2$$

$\Rightarrow x'$ is an upper bound $\Rightarrow \Leftarrow$

Case - III: $x_0^2 < 2$

{ Maybe we can avoid the
Lemma using an constructive
argument? }

Then choose $x' \in (x_0, \sqrt{2}) \cap \mathbb{Q}$

$$x^2 \leq (x_0)^2 < (x')^2 < 2$$

$\Rightarrow x' \in A \Rightarrow \Leftarrow$

TBD,
see
next
day

① The Real Number System

- Defⁿ $\mathbb{R}$ is the smallest totally ordered field containing $\mathbb{Q}$ as a subfield, that has lub property.

- Construction
 - Dedekind cuts
 - Cauchy Sequences Ⓢ
 - ...
- Out of Scope.

- The order on $\mathbb{R}$ is said to be "Dedekind-complete".

= (2023-08-31)

- Archimedean Property

$$\forall x \in \mathbb{R}, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{N} \mid nx > y$$

Proof: $A = \{nx \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$

Suppose not $\Leftrightarrow y$ is an upper bound of A .

$\because \mathbb{R}$ has sup property, $\exists \alpha \in \mathbb{R} = \sup A$.

$\because x > 0, \alpha = \sup A, \alpha - x$ is not an upper bound of A .

$$\Rightarrow \exists m \in \mathbb{N} \mid mx < \alpha - x \Rightarrow (m+1)x > \alpha \Rightarrow \text{contradiction} \quad (\because \alpha \text{ is an upper bound})$$

A is dense in S iff $\forall U \subseteq S$, $A \cap U \neq \emptyset$
 Equivalent as any open set in $\mathbb{R}$ contains an open interval.

• Theorem: $\mathbb{Q}$ is dense in $\mathbb{R}$.

$$\forall x < y ; x, y \in \mathbb{R} \quad \exists q \in \mathbb{Q} \mid x < q < y$$

NOTE: Implies the lemma used previously.

Proof: $x < y \Rightarrow y - x > 0$

By Archimedean Property, $\exists n \mid n(y-x) > 1$.

$$\text{Also } m_1(1) > nx \quad \& \quad m_2(1) > -nx$$

$\underbrace{m_1}_{n>0} \quad \underbrace{1}_{y} \quad \underbrace{m_2}_{n>0} \quad \underbrace{1}_{y}$

$$\therefore -m_2 < nx < m_1 \quad (\text{for some } n, m_1, m_2 \in \mathbb{N})$$

$$\Rightarrow m-1 \leq nx < m \quad (\text{for some integer } -m_2 \leq m \leq m_1)$$

Combining, we get:

$$nx < m \leq \frac{1+nx < ny}{1 < n(y-x)}$$

$$\Rightarrow x < \frac{m}{n} < y$$

$\underbrace{\frac{m}{n}}_{=: q}$

$$\Rightarrow \underline{x < q < y} \quad \blacksquare$$

• Corollary of Archimedean Property

By Archimedean Property: $A_x := \{n \in \mathbb{Z} \mid n > x\} \neq \emptyset$.

Now, $\forall n$ A_n has a lower bound.

↳ Observe $n \in A_n \Rightarrow n+1 \in A_n$

So if not, $A_n = \mathbb{Z}$

Claim: $\forall n, \exists m \in \mathbb{Z} \mid m \leq n$

If $x > 0$, $m = -1$, done.

If $x < 0$, $\exists m' \in \mathbb{N} \mid m'(1) > -x$

So, $\mathbb{Z} \setminus A_n \neq \emptyset$. $\Rightarrow \underbrace{-m'}_{=: m} < n$
done

$\mathbb{Z} \setminus A_n \subseteq \mathbb{R}$, & bounded above, $A_n \neq \emptyset$.

$\Rightarrow \exists! \alpha_n \in \mathbb{Z} \mid \alpha_n = \sup(\mathbb{Z} \setminus A_n)$

Define a f^n : $\lfloor x \rfloor := \alpha_n$

Hence, the "floor" function is well defined.

• Exercise: $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \quad \forall x \in \mathbb{R}$
HW

• $\lceil x \rceil = \begin{cases} x & , x \in \mathbb{Z} \\ \lfloor x \rfloor + 1 & , x \notin \mathbb{Z} \end{cases}$
"ceilings"

$\{x\} = x - \lfloor x \rfloor$ "frac"

⊙ Existence & Uniqueness of Decimal Expansion

<https://planetmath.org/existenceanduniquenessofdecimalexpansion>

A little too advanced, but read by yourself.

⊙ Uncountability of $\mathbb{R}$

(already done)

// • (Analog of Archimedean Property)

$$\forall x > 0, n \in \mathbb{N}, \exists! y \in \mathbb{R} \mid y^n = x.$$

(existence & uniqueness of n th roots)

$$y =: x^{1/n}$$

Can extend to the whole system of indices/powers.

// ⊙ (side quest) Continuum Hypothesis

Fact: $2^{\aleph_0} = \mathfrak{c}$

We know, $\mathbb{N} \cong \mathbb{Z} \cong \mathbb{Q} \cong \dots$ (countable sets)

$$\triangle \quad |\mathbb{N}| = \aleph_0 \text{ (aleph nought)}$$

And we know, $\mathbb{R}$ is uncountable, and, $|\mathbb{R}| = \mathfrak{c}$

Is there something between $\aleph_0$ & $\mathfrak{c}$? We can't know.

✓ Gödel's Incompleteness Theorem.