

Basic Metric Topology

ISI & CMI Yearlong Course 2024

Ⓐ Metric Spaces

⊙ Introduction

- Objective: To get a general framework for "distance", so that we can talk about convergence, Cauchy-ness etc
- Properties desired: For a "distance"
 - i) distance b/w 2 things can't be negative.
 - ii) 2 elements should have unique distance (well-defined)
 - iii) Symmetry: distance b/w A, B = distance b/w B, A .
 - iv) distance = 0 $\Leftrightarrow$ same elements.
- Def'n: Let X be a set. (X, d) is said to be a metric space, iff $d: X^2 \rightarrow \mathbb{R}_0^+$ | $\forall a, b \in X$...
 - axioms of metrics:
$$\begin{cases} d(a, b) \geq 0 \\ d(a, b) = 0 \Leftrightarrow a = b \\ d(a, b) = d(b, a) \quad \text{Symmetry} \\ d(a, b) \leq d(a, c) + d(c, b) \quad \Delta \text{ ineq.} \end{cases}, \forall c \in X$$
- ... where d is called the "metric".
- Eg: $(\mathbb{R}, d)$ where $d: (a, b) \mapsto |a - b|$
Euclidean metric on $\mathbb{R}$
Verify!

- Eg.: $(\mathbb{R}, d)$ where $d(a, b) = \mathbb{1}(a \neq b)$
discrete metric

- Eg.: (X, d) where $d(a, b) = \mathbb{1}(a \neq b)$

2) $(\mathbb{R}, d)$ where $d(a, b) = \mathbb{1}(a \neq b)$
 $\Rightarrow d(a, b) = \begin{cases} 0, & a=b \\ 1, & a \neq b \end{cases}$

i) $d(a, b)$ $(\mathbb{R}, d)$ where $d: \mathbb{R}^2 \rightarrow \{0, 1\}, \forall a, b \in \mathbb{R}$

a) $d(a, b) \geq 0, \forall a, b \in \mathbb{R}$ (Trivial)
 $d(a, b) = 0 \quad \forall a=b$
 $d(a, b) = 1 \quad \forall a \neq b$

b) $d(a, b) = 0 \Leftrightarrow a=b, \forall a, b \in \mathbb{R}$ (Trivial)
 $d(a, b) = \mathbb{1}(a \neq b) = 0 \quad \forall a=b$
 $\therefore d(a, b) = 0 \Leftrightarrow a=b \quad \forall a, b \in \mathbb{R}$

c) $d(a, b) = d(b, a) \quad \forall a, b \in \mathbb{R}$ (Trivial)
 Case 1: $(a=b)$
 $d(a, b) = 0$
 $d(b, a) = 0$
 $d(a, b) = d(b, a)$
 $\therefore d(a, b) = d(b, a) \quad \forall a, b \in \mathbb{R}$
 Case 2: $(a \neq b)$
 $d(a, b) = 1$
 $d(b, a) = 1$
 $d(a, b) = d(b, a)$
 $\therefore d(a, b) = d(b, a) \quad \forall a, b \in \mathbb{R}$

d) $d(a, b) \leq d(a, c) + d(c, b) \quad \forall a, b, c \in \mathbb{R}$
 $d(a, c) \in \{0, 1\}, d(c, b) \in \{0, 1\}$
 $(d(a, c) + d(c, b)) \in \{0, 1, 2\}$
 $\therefore d(a, b) \leq d(a, c) + d(c, b) \quad \forall a, b, c \in \mathbb{R}$

d) $d(a, b) \leq d(a, c) + d(c, b) \quad \forall a, b, c \in \mathbb{R}$ (Trivial)
 $d(a, c) \in \{0, 1\}$
 $d(c, b) \in \{0, 1\}$
 $d(a, b) \in \{0, 1\}$
 $(d(a, c) + d(c, b)) \in \{0, 1, 2\}$
 Case 1: $d(a, b) = 0$
 $\Leftrightarrow a=b$
 $d(a, b) \leq d(a, c) + d(c, b) \quad \forall c \in \mathbb{R}$
 Case 2: $d(a, b) = 1$
 $\Leftrightarrow a \neq b$
 If $d(a, c) + d(c, b) = 0$
 $\Rightarrow d(a, c) = d(c, b) = 0$
 $\Rightarrow a=c=b$
 (Contradiction)
 $\therefore \forall d(a, b) = 1, (d(a, c) + d(c, b)) \in \{1, 2\}$
 $\therefore d(a, b) \leq d(a, c) + d(c, b) \quad \forall a, b, c \in \mathbb{R}$

Notice, in the proof of (ii), we are only using the fact we can check $a=b$ in $\mathbb{R}$.
 $\therefore$ As long as we can do so in X , proof is same.

This teaches us a fundamental lesson: we can create structures based on a certain set of requirements they satisfy..., eg.: sets, magmas, monoids, groups, rings, fields, etc.

As long as we only use properties of a set, the proof will work for any set!

① Some Common Metric Spaces vector = $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$

- $(\mathbb{R}^n, d)$ where $d(\underline{a}, \underline{b}) = \|\underline{a} - \underline{b}\|$

$$\|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

This is called the Euclidean metric on $\mathbb{R}^n$.
(usual)

HW: Check this is a metric space.

- L_p norms on $\mathbb{R}^n$:

$$\|\underline{x}\|_p = |x_1^p + x_2^p + \dots + x_n^p|^{1/p}$$

Every norm induces a metric on $\mathbb{R}^n$:

$$d_p(\underline{a}, \underline{b}) = \|\underline{a} - \underline{b}\|_p.$$

Fact: $\forall p \geq 1$, d_p is a metric on $\mathbb{R}^n$

$p=1 \rightarrow$ Manhattan / taxi-cab distance

$p=2 \rightarrow$ Euclidean / straight line distance

$p=\infty \rightarrow$ max norm / max distance

$$\hookrightarrow \|\underline{x}\|_\infty = \max \{x_1, \dots, x_n\}$$

For $0 < p < 1$, triangle inequality is violated.

- L_p norms on $C^0(\mathbb{R})$

$C^0(\mathbb{R}) \rightarrow$ all continuous functions on $\mathbb{R}$.

$$d(f, g) = \|f - g\|_p$$

$$\|f\|_p = \left| \int_{\mathbb{R}} [f(x)]^p dx \right|^{1/p}$$

Here also, $p > 1 \Rightarrow$ metric, $0 < p < 1 \Rightarrow \otimes \Delta \text{ing.}$

- Fact: (X, d) metric space, $Y \subseteq X$.
 $\Rightarrow (Y, d)$ metric space.

- Neighbourhoods; Open & Closed Sets

- Defn: Let (X, d) be a metric space.

Let $x \in X$, $r > 0$. Then $B_r(x) \equiv B(x; r)$ is the r -neighbourhood of x .

$$B_r(x) = \{y \in X \mid d(x, y) < r\}$$

- Defn: Let (X, d) be a metric space, $E \subseteq X$, $x \in X$.
 Then x is called a limit point of E , iff.
 $\nexists r > 0, \exists y \neq x \in E \cap B_r(x)$
 Else, if $x \in E$, then x is called an isolated pt of E .
 iff. $\exists$ a sequence in $E \setminus \{x\}$, converging to x .

- Defⁿ: Let (X, d) be a metric space, $E \subseteq X$.
 $x \in X$ is said to be an....

- interior point of E , iff $\exists r > 0 \mid B_r(x) \subseteq E$
- exterior point of E , iff $\exists r > 0 \mid B_r(x) \subseteq E^c$
- boundary point of E , iff $\forall r > 0, B_r(x) \cap E \neq \emptyset$
 $B_r(x) \cap E^c \neq \emptyset$

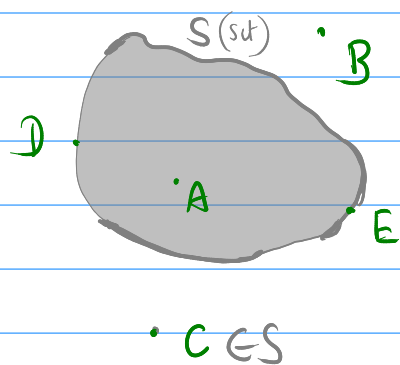
- Notations Let (X, d) be a metric space, $E \subseteq X$.

$E^\star =$ set of all limit points of E .

why? $E \setminus E^\star =$ set of all isolated points of E .

$E^\circ = \text{Int}(E) =$ set of all interior points of E
 $\text{ext}(E) =$ " exterior "
 $\partial E =$ " boundary "

- So, now we have 2 classification criteria which are mutually exclusive & exhaustive.
 Time visualise the two & relate them.



- A $\rightarrow$ interior, limit
- B $\rightarrow$ exterior, (neither limit, ^{nor} boundary)
- C $\rightarrow$ boundary, isolated ($\notin S$)
- D $\rightarrow$ boundary, limit, $\notin S$
- E $\rightarrow$ boundary, limit, $\in S$

Drawing convention:



boundary excluded



boundary included (in bold)

	Limit, $\in S$	Limit, $\notin S$	$\hookrightarrow ES$ implied Isolated	Not limit & $\notin S$
Interior	$\checkmark (A)$	$\times$	$\bullet$	$\times$
Exterior	$\times$	$\times$	$\times$	$\checkmark (B)$
Boundary	$\checkmark (E)$	$\checkmark (D)$	$\checkmark (C)$	$\times$

$\checkmark \leftarrow$ possible
 $\times \leftarrow$ impossible
 $\bullet \leftarrow$ possible using discrete metric
 $\rightarrow$ examples given.

$\hookrightarrow$ discrete metric $\Rightarrow$ no convergence!
 Isolated: $\nexists$ seq. converging to 0.
 Interior: $B_{\frac{1}{2}}(0,1) = \{0\} \subseteq \{0,1,2\}$
 $X = \mathbb{Z}$, $d = 1$ ($a \neq b$)
 Then, 0 is an isolated interior point of $\{0,1,2\}$

• Results $E \subseteq X$. $\text{int}(X) = \text{ext}(X^c)$
 $\partial X = \partial(X^c)$

• Defⁿ: A set $E \subseteq X$ is said to be open iff all points are interior points, i.e. $E = E^\circ$, $\exists \varepsilon_i (0,1) \text{ in } \mathbb{R}$

• Defⁿ: A set $E \subseteq X$ is said to be closed iff it includes all its limit points, i.e. $E^* \subseteq E$, $\exists \varepsilon_i [0,1] \text{ in } \mathbb{R}$

• Defⁿ: A set $E \subseteq X$ is said to be perfect iff $E \subseteq E^*$, $\exists \varepsilon_i \mathbb{Q} \text{ in } \mathbb{R}$

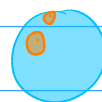
• Defⁿ: $\overline{E} := E \cup E^* (= E \cup \partial E)$ why? (HW)
 $\hookrightarrow$ is called the closure of E , ($\overline{\mathbb{Q}} = \mathbb{R}$)

• Results: $\forall E \subseteq X$, E° is open, $\overline{E}$ is closed.
 $\forall E \subseteq X$, $\overline{(E^\circ)} = \overline{E}$, $(\overline{E})^\circ = E^\circ$

• Result: E open in $X \iff E^c$ closed in X .

• Defn E is dense in X iff $E \cup E^{\Delta} = X$.
Eg.: $\mathbb{Q}$ is dense in $\mathbb{R}$.

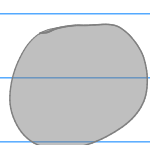
• Result Every neighbourhood is open

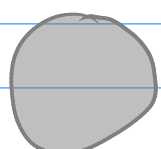


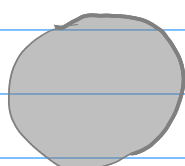
• Result: A finite set has no limit points.

⑥ HCU Compute $\partial\mathbb{Q}$, $\mathbb{Q}^{\Delta}$ (Both should be $\mathbb{R}$)

• Examples

 $\leftarrow$ open (doesn't include boundary)

 $\leftarrow$ closed (includes full boundary)

 $\leftarrow$ neither open nor closed.

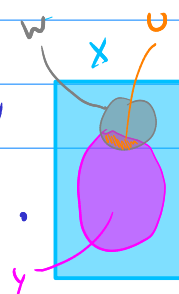
• Result: $\partial S \subseteq S \iff S$ is closed.
 $\partial S \subseteq S^c \iff S$ is open

(So, for us, it holds)

does hold under a continuous metric.

$\rightarrow$ no! (isolated interior points exist)

• Result: (X, d) metric space, $Y \subseteq X$; $U \subseteq Y$
 Then $U = W \cap Y$ for some $W \subseteq X$.



2.21 Examples Let us consider the following subsets of $\mathbb{R}^2$:

- (a) The set of all complex z such that $|z| < 1$.
- (b) The set of all complex z such that $|z| \leq 1$.
- (c) A nonempty finite set.
- (d) The set of all integers.
- (e) The set consisting of the numbers $1/n$ ($n = 1, 2, 3, \dots$). Let us note that this set E has a limit point (namely, $z = 0$) but that no point of E is a limit point of E ; we wish to stress the difference between having a limit point and containing one.
- (f) The set of all complex numbers (that is, $\mathbb{R}^2$).
- (g) The segment (a, b) .

Let us note that (d), (e), (g) can be regarded also as subsets of $\mathbb{R}^1$. Some properties of these sets are tabulated below:

	<i>Closed</i>	<i>Open</i>	<i>Perfect</i>	<i>Bounded</i>
(a)	No	Yes	No	Yes
(b)	Yes	No	Yes	Yes
(c)	Yes	No	No	Yes
(d)	Yes	No	No	No
(e)	No	No	No	Yes
(f)	Yes	Yes	Yes	No
(g)	No		No	Yes

In (g), we left the second entry blank. The reason is that the segment (a, b) is not open if we regard it as a subset of $\mathbb{R}^2$, but it is an open subset of $\mathbb{R}^1$.

Use Rudin as primary reference for Topology.

Q Let $G \subseteq E \subseteq X$, G is open.
Prove: $G \subseteq E^\circ$.

∂S
 $= \bar{S} \setminus S^\circ$
 holds

Ans! Suppose not. $G \subseteq E \subseteq \bar{E} = E^\circ \cup \partial E$
 $\therefore G \not\subseteq E^\circ, \exists x \in G \cap \partial E$

$x \in \partial E \Rightarrow \forall r > 0, B_r(x) \cap E^c \neq \emptyset$ (by defⁿ)

Now, $G \subseteq E \Rightarrow E^c \subseteq G^c \Rightarrow \forall r > 0, B_r(x) \cap G^c \neq \emptyset$ — ①

Now, $x \in G \Rightarrow \forall r > 0, B_r(x) \cap G \neq \emptyset$. ② $\therefore x \in \partial G$

But, $G = G^\circ$ ($\because G$ open), & $G^\circ \cap \partial G = \emptyset \Rightarrow \Leftarrow$

from ① & ②

⑥ Completeness

- Defⁿ: A metric space (X, d) is said to be complete, iff every Cauchy sequence converges
- Eg¹: $\mathbb{R}$ is complete (proof is out of scope)
 $\mathbb{Q}$ is not complete (proof done earlier)
... under the usual metric.

⑥ Let (X, d) be a metric space, $d(a, b) = 1$ ($a \neq b$)
Show this is complete.

Ans) Let $\{x_n\}$ be a Cauchy sequence in X .

Claim: $\{x_n\}$ has a constant tail. (say $= x$)
 $\exists N \in \mathbb{N} \mid \forall n \geq N, x_n = x$

If the above claim holds, then obviously, $x_n \rightarrow x$.
(in fact, convergence in discrete metric $\Leftrightarrow$ constant tail)

$\therefore \{x_n\}$ Cauchy, $\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \forall m, n \geq N, d(x_n, x_m) < \varepsilon$
 $\underbrace{1(x_n \neq x_m)}$

Take $\varepsilon = 1/2$.

$\therefore \exists N \in \mathbb{N} \mid \forall m, n \geq N, 1(x_n \neq x_m) < 1/2$

$\therefore \exists N \in \mathbb{N}$ after which
sequence becomes constant. $\parallel \begin{cases} \Leftrightarrow 1(x_n \neq x_m) = 0 \\ \Leftrightarrow x_n = x_m \end{cases}$

Hence, convergent. ■

- What does a complete set look like? (in $\mathbb{R}^2$)

We are already familiar with the idea of sequences "wanting to converge", i.e. Cauchy. The terms get more and more tightly packed around something... which we now know, is a limit point of the set (or is it)?

Can you always guarantee the existence of this limit point?

Consider the usual metric on $\mathbb{R} \setminus \{0\}$.

The sequence $\{\frac{1}{n}\}$ is Cauchy in $(0,1]$ (which is closed in $\mathbb{R} \setminus \{0\}$)

But, $\frac{1}{n} \rightarrow 0 \notin (0,1]$ $\therefore (0,1]$ is not complete in $\mathbb{R} \setminus \{0\}$.

But observe $(0,1]$ is closed in $\mathbb{R} \setminus \{0\}$.

$\therefore 0 \notin \mathbb{R} \setminus \{0\}$, it is not a limit point of $(0,1]$.
(in $\mathbb{R} \setminus \{0\}$)

However, if there are no "holes" in X , one may guess
closed $\Leftrightarrow$ complete

- Result: $\mathbb{R}^n$ is complete $\forall n$.

- Result: In a complete metric space (X, d)
 $E \subseteq X$ closed $\Leftrightarrow (E, d)$ complete

Using this, $\mathbb{R}$ is complete, but $\mathbb{R} \setminus \{0\}$ is not closed in $\mathbb{R}$
 $\therefore$ Not complete!

- Intuitively, "closed" & "complete" mean the same thing, as long as we can work around the existence of limit points in the parent space.

Completeness doesn't care about whether the limit point actually exists! It just finds a hole where a Cauchy sequence can fall into.

but no convergence? Must be a hole!

Closed sets (thru limit points) are just a way to classify points in X , w.r.t some $E \subseteq X$.

$\hookrightarrow$ limit $\in E$, limit $\notin E$, isolated ($\in E$), $\notin E$ & not limit

Crucially, we only classify points in X .

- In fact, this is a powerful property of Cauchy sequences. They find "holes" in your metric space.

You can "fill" those "holes" to produce the completion of a metric space.

Eg: completion of $\mathbb{Q}$ is $\mathbb{R}$.

$\hookrightarrow$ alternate formulation of real numbers!

no mention of ordering.

- We couldn't have done the same thing using closures, as $\mathbb{Q}$ is closed in $\mathbb{Q}$, we would need to know about $\mathbb{R}$ already... *(in fact...)*

- Result: Let (X, d) be a metric space.
 - X is open in X
 - X is closed in X .

Corollary: Taking complement, ϕ is also both open & closed in X .
~~(in fact, only X, ϕ are clopen)~~ *(closed & open.)*

Proof:

$$i) \left. \begin{array}{l} \forall x \in X, \forall r > 0, B_r(x) \subseteq X \\ (\because X \text{ is the metric space}) \end{array} \right\} \Rightarrow X \text{ is open}$$

ii) $X^* \subseteq X$ $\because$ X is the metric space, we can only classify points in the space...

You can't consider points outside the metric space for limit points & neighbourhoods!

© Banach Contraction Mapping Theorem

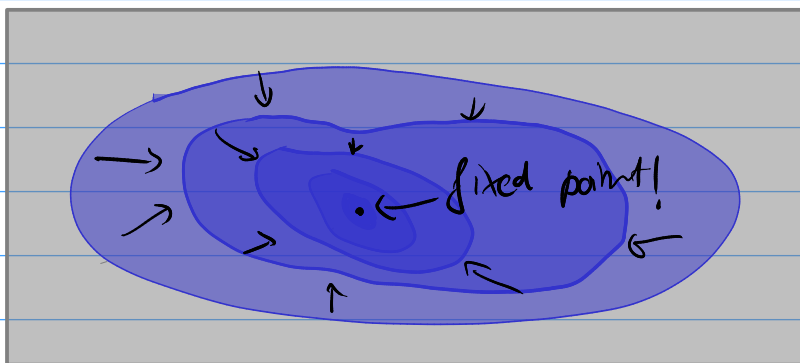
- Defⁿ: $f: X \rightarrow X$ is said to be a contraction map if $\exists c \in [0, 1) \mid d(f(a), f(b)) \leq c d(a, b) \quad \forall a, b \in X$
- Defⁿ: If $f(x) = x$, x is called a fixed point of f .

A B

Theorem In a non-empty complete metric space, a contraction map has exactly one fixed point.

(Proof: Wikipedia)

Essence of the proof is a geometric argument.



Def'n: $f: A \rightarrow X$, $d(f(a), f(b)) \leq k d(a, b) \forall a, b \in A$
 Then, f is said to be Lipschitz on A .
 (Lipschitz constant k)

Contraction map $\Rightarrow$ Lipschitz

Connectedness

Def'n: 2 sets $A, B \subseteq X$ are said to be separated iff $\bar{A} \cap B = \emptyset$ & $A \cap \bar{B} = \emptyset$.
 (they don't intersect each other's closures)

Observation: Separated $\not\Rightarrow$ Disjoint
 $[0, 1), (1, 2]$ $\checkmark$ $\rightarrow$ $[0, 1), [1, 2]$

Def'n: A set $E \subseteq X$ is called separable or disconnected iff $\exists A, B \subseteq X$ separated & non-empty | $E = A \cup B$.

A set $E \subseteq X$ is called connected iff it is not separable.
 Eg: any interval in $\mathbb{R}$.

- Result: TFAE (the following are equivalent)

i) X separable (i.e. disconnected)

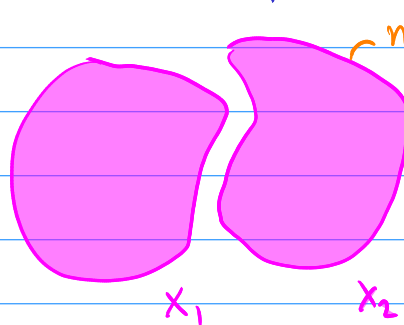
ii) $X = A \cup B$ where $A, B \subseteq X$.

iii) $E \subseteq X$ clopen $\Rightarrow E = \emptyset$ or $E = X$. } was talking about before.

iv) $E \subseteq X, \partial E = \emptyset \Rightarrow E = \emptyset$ or $E = X$

$\Rightarrow X$:

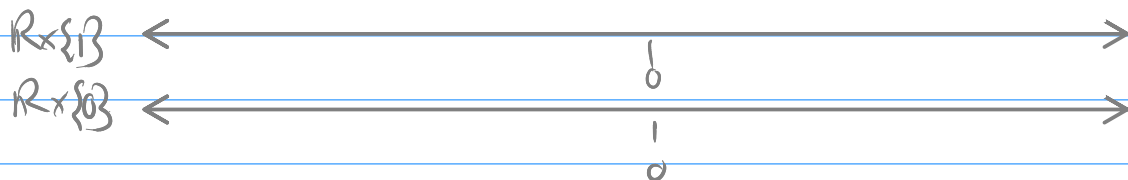
How Check!



$$\partial X_1 = \partial X_2 = \emptyset$$

X_1, X_2 are clopen in X as any points "outside" are not in X !

Another example: 2 real lines.



- Connected set in $\mathbb{R} \Leftrightarrow$ interval

⊙ Compactness

(Reference: Readin)

$\hookrightarrow$ wrt X

- Defⁿ: A collection of open sets $\{G_\alpha\}_{\alpha \in A}$ is called an open cover on $E \subseteq X$ iff $E \subseteq \bigcup_{\alpha \in A} G_\alpha$

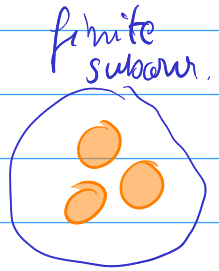
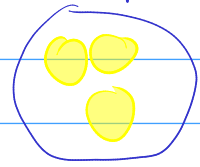
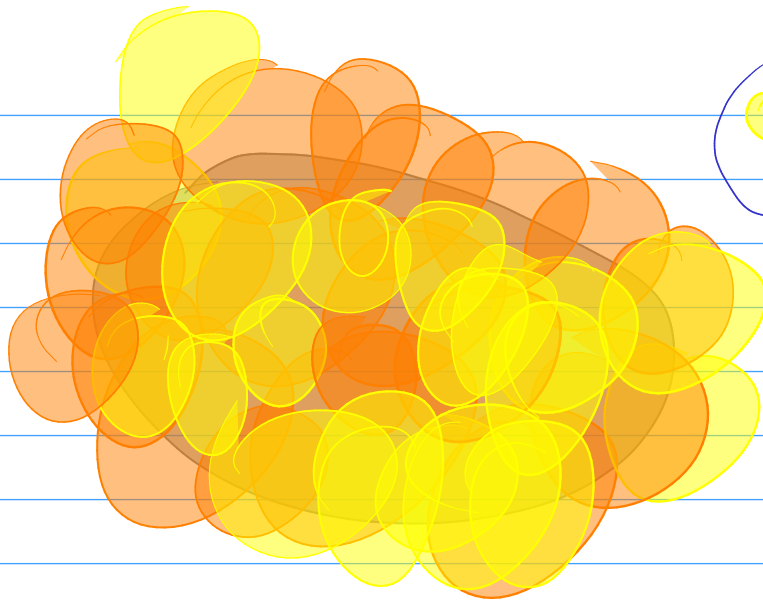
- Defⁿ: $K \subseteq X$ is called compact iff every open cover of K has a finite subcover, i.e. $\exists \alpha_1, \dots, \alpha_n \in A$

Every infinite open cover "overcovers" K

$$E \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$$

possibly infinitely many sets!
open cover

- The Idea



- Result: $K \subseteq Y \subseteq X$ is compact wrt X ~~iff~~ it is compact wrt Y

- Result: $K \subseteq X$ compact $\Rightarrow K$ closed & bounded.

- Result: $K \subseteq X$ compact. $E \subseteq K$ closed. Then E compact.

- Result: Every cartesian product of k closed & bounded intervals is compact in $\mathbb{R}^k$. ↳ k -cell

- Heine-Borel Theorem

In $\mathbb{R}^n$, K compact $\Leftrightarrow$ K closed & bounded.

(proof is out of scope)

working defⁿ for compactness.

- Watch this: <https://www.youtube.com/watch?v=td7Nz9ATyWY>

// END